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On the Normalization and Hermiticity of Amplitudes in 4D Heterotic Superstrings

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ABSTRACT

We consider how to normalize the scattering amplitudes of 4D heterotic superstrings in a Minkowski background. We fix the normalization of the vacuum amplitude (the string partition function) at each genus, and of every vertex operator describing a physical external string state in a way consistent with unitarity of the S-matrix. We also provide an explicit expression for the map relating the vertex operator of an incoming physical state to the vertex operator describing the same physical state, but outgoing. This map is related to hermitean conjugation and to the hermiticity properties of the scattering amplitudes.

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Introduction and Summary

String theory [1] remains the most promising candidate for a quantum theory of gravity. It has also proven itself useful as a tool for perturbative calculations in Yang-Mills theory [2]. Accordingly, it is of interest to be able to make detailed computations of string scattering amplitudes at any loop order. It is well known how to do this by means of the Polyakov path integral or, equivalently, by computing vacuum expectation values: There exists a “master formula” expressing the connected part of the scattering amplitude at each loop level as an integral over moduli space, where the integrand is obtained as a correlation function of vertex operators, with the appropriate insertions of world-sheet ghosts and Picture Changing Operators (PCOs) [3]. In order to obtain from this “master formula” an actual scattering amplitude (i.e. a number) one would have to perform the integral over the moduli as well as the summation over spin structures, both of which are usually impossible by analytical means.

In this paper we address two other important points that have to be understood in detail to be able to obtain explicit expressions for string scattering amplitudes.

First of all, we need to know what is the correct normalization of the vacuum amplitude (the string partition function) at each and every genus, and also what is the normalization of all the vertex operators describing physical external string states. This is obviously an important issue since, for example, it is through the proper normalization of the vertex operators that there appears the relation between the string length scale parameter α', the gravitational coupling constant κ and the gauge coupling constants. In a second quantized theory, like Quantum Field Theory, the proper normalizations are obtained automatically when computing the amplitudes using, for example, Dyson’s formula. Instead, in the first quantized framework of string theory, one has to carefully fix all normalizations in a way consistent with unitarity of the S-matrix.

Second, we need to understand what is the *exact* relation between the vertex operators that we use to describe ingoing and outgoing string states in the “master formula”. We

may formulate this more precisely: Consider some scattering process,

$$\lambda_{N_{out}+N_{in}} + \ldots + \lambda_{N_{out}+1} \;\longrightarrow\; \lambda_{N_{out}} + \ldots + \lambda_1 \ , \tag{1}$$

where each label λ represents a set of single-string state quantum numbers, such as momentum, helicity, charges etc. By definition the quantum mechanical scattering amplitude $A_{f\leftarrow i}$ for this process is given by the S matrix element

$$A_{f\leftarrow i} \;=\; \langle \lambda_1, \ldots, \lambda_{N_{out}}; in|S|\lambda_{N_{out}+1}, \ldots, \lambda_{N_{out}+N_{in}}; in\rangle \ , \tag{2}$$

that involves only "*in*" states. Incoming strings are described by ket-states, outgoing ones by bra-states.

In string theory we may compute the connected part of this transition amplitude by means of the "master formula", where each single-string state —whether appearing in eq. (2) as a bra or a ket— is represented by a vertex operator. The question is the following: If some vertex operator $\mathcal{W}_{|\lambda\rangle}$ represents the single-string ket-state $|\lambda; in\rangle$, what is the vertex operator $\mathcal{W}_{\langle\lambda|}$ that represents the single-string bra-state $\langle\lambda; in|$?

Since $\langle\lambda; in| = (|\lambda; in\rangle)^\dagger$ it is clear that this question is closely related to the hermiticity properties of the scattering amplitude: The correct choice of $\mathcal{W}_{\langle\lambda|}$ should lead to S-matrix elements consistent with unitarity. In particular it should lead to tree-level T-matrix elements that are real away from the momentum poles.

In field theory, in the setting of the Lehmann-Symanzik-Zimmermann reduction formula for S-matrix elements, $\mathcal{W}_{\langle\lambda|}$ is just the hermitean conjugate of $\mathcal{W}_{|\lambda\rangle}$. In string theory, due primarily to the presence of PCOs in the "master formula", the relation between $\mathcal{W}_{\langle\lambda|}$ and $\mathcal{W}_{|\lambda\rangle}$ turns out to be somewhat modified.

In practice the two problems, finding the correct normalization of the vertex operators appearing in the "master formula", and deriving the exact relation between $\mathcal{W}_{\langle\lambda|}$ and $\mathcal{W}_{|\lambda\rangle}$, can be solved at the same time. We could imagine considering the connected part of the tree-level two-point amplitude (i.e. the inverse propagator) and impose that this should assume the canonical form known from field theory. But the "master formula" for connected string theory amplitudes is only well-defined on the mass shell and here the inverse propagator vanishes identically.

Instead we consider another simple object, which is nonzero even on-shell and just as universal as the propagator. This is the amplitude for any given string state to emit or absorb a zero-momentum graviton without changing any of its own quantum numbers.

More precisely we consider the term that describes the universal coupling of gravity to the 4-momentum of the propagating string state and the requirement that this term assumes its canonical form yields not only an expression for the normalization of the vertex operator in terms of the gravitational coupling κ (which in $D = 4$ dimensions is related to Newton's constant by $\kappa^2 = 8\pi G_N$), it also provides the desired map between the vertex operators $\mathcal{W}_{|\lambda\rangle}$ and $\mathcal{W}_{\langle\lambda|}$.

The procedure that we adopt is a development of the method proposed in ref. [4], where it was suggested to normalize the vertex operator of any given string state by considering the elastic scattering of this string state, and some "reference" string state, for example a graviton, at very high center-of-mass energies, where the interactions are dominated by gravity, and require that the tree-level amplitude for this process reproduces the standard one dictated by the principle of equivalence. But whereas in ref. [4] the method of normalization was only applied to a few examples, in this paper we proceed to find the proper normalization for all vertex operators in the string theory.

The relation between the vertex operators $\mathcal{W}_{|\lambda\rangle}$ and $\mathcal{W}_{\langle\lambda|}$ and the associated question of unitarity of the S-matrix was also discussed in ref. [5]. We provide an explicit expression for $\mathcal{W}_{\langle\lambda|}$, including an overall phase factor, which depends on the picture of the vertex operator, that was not manifest in ref. [5].

The paper is organized as follows: In section 1 we review the situation in quantum field theory, where the Lehmann-Symanzik-Zimmermann reduction formula involves different operators for incoming and outgoing particles, in analogy with the situation we encounter in string theory. In section 2 we present the "master formula" for string amplitudes and section 3 contains a discussion of the correct overall normalization of the vacuum amplitude. In Section 4 we obtain an ansatz for the map between $\mathcal{W}_{|\lambda\rangle}$ and $\mathcal{W}_{\langle\lambda|}$, which is subsequently verified in section 5, where the normalization of the vertex operators is also derived. In Section 6 we check that our ansatz is consistent with unitarity in the sense that it leads to real tree-level amplitudes away from the momentum poles. Section 7 contains an explicit example in the framework of four-dimensional heterotic string theories built using free world-sheet fermions. Finally we include two appendices containing various conventions and a third appendix devoted to the proof of the compatibility of the GSO projection and the map between $\mathcal{W}_{|\lambda\rangle}$ and $\mathcal{W}_{\langle\lambda|}$.

1. Field theory

We can formulate scattering amplitudes in 4D field theory in a form close to the one we use in string theory by means of the Lehmann-Symanzik-Zimmermann reduction formula for S-matrix elements [6]:

$$\begin{aligned}\langle\lambda_1,\ldots,\lambda_{N_{out}};in|S|\lambda_{N_{out}+1},\ldots,\lambda_{N_{out}+N_{in}},;in\rangle \;=\; \text{disconnected terms} + \qquad (1.1)\\ \prod_{j=1}^{N_{out}+N_{in}}\left(\frac{i}{\sqrt{Z_j}}\right)\int\prod_{j=1}^{N_{out}+N_{in}}\left(\mathrm{d}^4x_j\right)\langle 0|\mathrm{T}V_{\langle\lambda_1|}(x_1)\ldots V_{|\lambda_{N_{out}+N_{in}}\rangle}(x_{N_{\mathrm{out}}+N_{in}})|0\rangle\ .\end{aligned}$$

Here we have a Field Theory Vertex (FTV) $V_{|\lambda\rangle}(x)$ corresponding to the 1-particle ket-state $|\lambda;in\rangle$ where the label λ incorporates the 4-momentum p as well as other quantum numbers, and similarly we have a FTV $V_{\langle\lambda|}(x)$ corresponding to the 1-particle bra-state $\langle\lambda;in|$.

Since by definition of hermitean conjugation $\langle\lambda;in| = (|\lambda;in\rangle)^\dagger$, it is not surprising that $V_{\langle\lambda|}(x)$ is just the hermitean conjugate of $V_{|\lambda\rangle}(x)$,

$$V_{\langle\lambda|}(x) = \left(V_{|\lambda\rangle}(x)\right)^\dagger\ . \qquad (1.2)$$

For example, for a particle described by a real scalar field ϕ, the 1-particle states are specified by their momentum only and the Field Theory Vertices are

$$\begin{aligned}V_{|p\rangle}(x) =&\ e^{ip\cdot x}\left(-\Box_x+m^2\right)\phi(x) \qquad (1.3)\\ V_{\langle p|}(x) =&\ e^{-ip\cdot x}\left(-\Box_x+m^2\right)\phi(x)\ ,\end{aligned}$$

where in both cases $p^0 = +\sqrt{\vec{p}^{\,2}+m^2}$ and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ with $\eta = \mathrm{diag}(-1,1,1,1)$.

Another example is provided by an electron with momentum p and helicity η, where

$$\begin{aligned}V_{|e^-,p,\eta\rangle}(x) =&\ -\overline{\psi}(x)\left(\overleftarrow{\not\partial}_x - m\right)u(\vec{p},\eta)e^{ip\cdot x} \qquad (1.4)\\ V_{\langle e^-,p,\eta|}(x) =&\ -\overline{u}(\vec{p},\eta)\left(-\not\partial_x-m\right)\psi(x)e^{-ip\cdot x}\ .\end{aligned}$$

Here $\overline{\psi} = \psi^\dagger(i\gamma^0) = \psi^\dagger(-i\gamma_0)$ and $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$. The spinor $u(\vec{p},\eta)$ of the incoming particle with momentum p and helicity η satisfies the Dirac equation $(i\not p - m)u(\vec{p},\eta) = 0$ and is normalized according to

$$u^\dagger(\vec{p},\eta)u(\vec{p},\eta') = 2p^0\delta_{\eta,\eta'}\ . \qquad (1.5)$$

For particles of nonzero mass m this normalization is equivalent to the more standard one $\overline{u}(\vec{p},\eta)u(\vec{p},\eta') = 2m\delta_{\eta,\eta'}$, but unlike the standard normalization condition it can also be used for massless particles.

2. String amplitudes

In this paper we only consider 4D heterotic string models in a Minkowski background. We define the T-matrix element as the connected S-matrix element with certain normalization factors removed

$$\frac{\langle\lambda_1;\ldots;\lambda_{N_{out}};in|S|\lambda_{N_{out}+1},\ldots,\lambda_{N_{out}+N_{in}};in\rangle_{\text{connected}}}{\prod_{i=1}^{N_{tot}}\left(\langle\lambda_i;in|\lambda_i;in\rangle\right)^{1/2}} = \tag{2.1}$$
$$i(2\pi)^4\delta^4(p_1+\ldots+p_{N_{out}}-p_{N_{out}+1}-\ldots-p_{N_{tot}})\prod_{i=1}^{N_{tot}}(2p_i^0V)^{-1/2}\times$$
$$T(\lambda_1;\ldots;\lambda_{N_{out}}|\lambda_{N_{out}+1};\ldots;\lambda_{N_{out}+N_{in}})\ ,$$

where $N_{tot} = N_{in} + N_{out}$ is the total number of external states, p_i is the momentum of the i'th string state, all of them having $p_i^0 > 0$, and V is the usual volume-of-the-world factor. We also introduce the dimensionless momentum $k_\mu \equiv \sqrt{\frac{\alpha'}{2}}\,p_\mu$. The Minkowski metric is $\eta = \text{diag}(-1,1,1,1)$.

For heterotic superstrings in the Neveu-Schwarz Ramond formalism we have various free conformal fields: The space-time coordinates X^μ, their chiral world-sheet superpartners ψ^μ, the reparametrization ghosts b,c and $\bar{b},\bar{c}$, and the superghosts β,γ. On top of this we have various internal degrees of freedom described by a conformal field theory (CFT) with left-moving (right-moving) central charge 22 (9). These may or may not be free. The g-loop contribution to the T-matrix element is given by the Polyakov path integral which is equivalent to the following operator formula

$$T^g(\lambda_1;\ldots;\lambda_{N_{out}}|\lambda_{N_{out}+1};\ldots;\lambda_{N_{out}+N_{in}}) = \tag{2.2}$$
$$(-1)^{g-1}C_g\int\prod_{I=1}^{3g-3+N_{tot}}(\mathrm{d}^2m^I)\prod_{\mu=1}^{g}\left(\sum_{\boldsymbol{\alpha}_\mu,\boldsymbol{\beta}_\mu}C\begin{matrix}\boldsymbol{\alpha}_\mu\\ \boldsymbol{\beta}_\mu\end{matrix}\right)\langle\langle\left|\prod_{I=1}^{3g-3+N_{tot}}(\eta_I|b)\prod_{i=1}^{N_{tot}}c(z_i)\right|^2\times$$
$$\left(\prod_{A=1}^{2g-2+N_B+N_{FP}}\Pi(w_A)\right)\mathcal{V}_{\langle\lambda_1|}(z_1,\bar{z}_1)\ldots\mathcal{V}_{|\lambda_{N_{tot}}\rangle}(z_{N_{tot}},\bar{z}_{N_{tot}})\rangle\rangle\ .$$

Here C_g is a constant giving the proper normalization to the string partition function (the g-loop vacuum amplitude). It will be given explicitly in section 3, and (as we shall see) the sign $(-1)^{g-1}$ ensures that C_g is a positive number. m^I is a modular parameter, η_I is the corresponding Beltrami differential, and our conventions for the overlap $(\eta_I|b)$ with the antighost field b are defined in detail in ref. [7]. The integral is over one fundamental domain of N_{tot}-punctured genus g moduli space. For each loop, labelled by $l = 1, \ldots, g$, we have a summation over sets of spin structures, collected in vectors $\boldsymbol{\alpha}_l$ and $\boldsymbol{\beta}_l$, and with a summation coefficient $C^{\boldsymbol{\alpha}_l}_{\boldsymbol{\beta}_l}$. By definition the correlator $\langle\langle \ldots \rangle\rangle$ includes the partition function. At tree level, where the non-zero mode partition function is equal to one, the notation $\langle \ldots \rangle$ is also used. At loop level we choose the normalization for the partition function to be the one obtained by applying the sewing procedure. This guarantees sensible factorization properties in the corner of moduli space where the world-sheet degenerates into individual tori connected by long tubes and implies that the spin-structure summation coefficient is just a product of one-loop summation coefficients as in eq. (2.2). More details on our conventions for spin structures, partition functions and operator fields in the explicit setting of a heterotic string model built with free world-sheet fermions [8,9,10] can be found in Appendix A, see also refs. [11,12].

In analogy with field theory we have introduced a vertex operator $\mathcal{V}_{|\lambda\rangle}(z,\bar{z})$ for each ket string state $|\lambda\rangle$ and similarly a vertex operator $\mathcal{V}_{\langle\lambda|}(z,\bar{z})$ corresponding to each bra string state $\langle\lambda|$. [1] At the end of this section we will have more to say about the meaning of these operators.

The ghost factors residing in the BRST invariant version of the vertex operator, given by

$$\mathcal{W}_{|\lambda\rangle}(z,\bar{z}) \; = \; c(z)\overline{c}(\bar{z})\mathcal{V}_{|\lambda\rangle}(z,\bar{z}) \qquad \text{and} \qquad \mathcal{W}_{\langle\lambda|}(z,\bar{z}) \; = \; c(z)\overline{c}(\bar{z})\mathcal{V}_{\langle\lambda|}(z,\bar{z}) \; , \tag{2.3}$$

have been factored out in eq. (2.2). We take all space-time bosonic vertex operators to be in the $q = -1$ superghost picture and all the space-time fermionic vertex operators to be in the $q = -1/2$ superghost picture. In an amplitude involving N_B space-time bosons and $2N_{FP}$ space-time fermions this implies that we have to insert $2g - 2 + N_B + N_{FP}$ PCOs Π at arbitrary points w_A on the Riemann surface. In practical calculations it can be convenient to insert one PCO at each of the vertex operators describing the space-time

[1] Since all the states we consider are of the "*in*" variety, we drop the "*in*" label from now on.

bosons so as to change these into the $q = 0$ picture. This leaves $2g - 2 + N_{FP}$ PCOs at arbitrary points.

If we "bosonize" the superghosts in the usual way, $\beta = \partial\xi e^{-\phi}$ and $\gamma = e^{+\phi}\eta$, the PCO is given explicitly by

$$\Pi = 2c\partial\xi + 2e^{\phi}T_F^{[X,\psi]} - \frac{1}{2}\partial(e^{2\phi}\eta b) - \frac{1}{2}e^{2\phi}(\partial\eta)b \ , \tag{2.4}$$

where we suppressed the superghost cocycle factor which ensures that e^{ϕ} anti-commutes with all other fermionic operators on the world-sheet, and

$$T_F^{[X,\psi]} \ = \ -\frac{i}{2}\partial X \cdot \psi + (\text{internal part}) \tag{2.5}$$

is the orbital part of the world-sheet supercurrent (i.e. the part not involving ghosts and superghosts). The "internal part" refers to the internal right-moving degrees of freedom of the CFT with central charge 9.

As stated in the introduction our aim in this paper is twofold: First, since the T-matrix element as defined in eq. (2.1) corresponds to the connected S-matrix element obtained using states with standard field theory normalization, we have to use vertex operators with a definite normalization in eq. (2.2). So we need to know what is the correct normalization of all vertex operators involved in the theory; and we also need to determine the value of the overall normalization constant C_g. Second, we need to understand what is the *exact* relation between the vertex operators $\mathcal{W}_{\langle\lambda|}(z,\bar{z})$ and $\mathcal{W}_{|\lambda\rangle}(z,\bar{z})$. By definition the operator $\mathcal{W}_{|\lambda\rangle}(z=0)$, when acting on the conformal vacuum $|0\rangle$, creates the string state $|\lambda\rangle$, where (like in section 1) λ is a label incorporating the 4-momentum k (with $k^0 > 0$), the helicity and the "particle type" (defined through the values of various charges and family labels). We may think of eq. (2.2) as an indirect definition of what we mean by $\mathcal{W}_{\langle\lambda|}$: It is the vertex operator we have to use on the right-hand side of this equation in order to obtain the T-matrix element involving the bra-state $\langle\lambda| = (|\lambda\rangle)^\dagger$. Of course this definition is somewhat circular, because we don't know how to compute the T-matrix element until we have specified what are the vertex operators. Indeed, as explained in the introduction, the procedure we adopt is to carefully *derive* what $\mathcal{W}_{\langle\lambda|}$ should be in order for the "master formula" (2.2) to reproduce the correct amplitude for a propagating string to emit or absorb a zero-momentum graviton. Based on our experience from field theory, as outlined in section 1, we might expect $\mathcal{W}_{\langle\lambda|}$ to be given by the hermitean conjugate of $\mathcal{W}_{|\lambda\rangle}$. As we shall see in section 4, this is not completely correct.

3. Normalization of the vacuum amplitude

In section 2 we already made use of the basic fact that the problem of normalizing string amplitudes can be separated into two independent problems: One, to fix the normalization constant C_g of the vacuum amplitude at genus g. The other, to fix the normalization of each vertex operator in the theory.

It is factorization that leads to this simple result. For example, to see that the normalization of the vertex operators cannot depend on the topology of the world-sheet we can imagine inserting a vertex operator on a sphere connected by a long tube to some genus g surface. It is clear that the vertex operator cannot know about the distant handles. This is true even for vertex operators describing space-time fermions, even though these involve spin fields which are non-local operators on the world-sheet, because space-time fermions always come in pairs and we may imagine isolating both of the corresponding vertex operators (and the branch cut connecting them) on a sphere far away from all handles.

Similarly, if we assume for the moment that the overall normalization of the amplitude depends on the number N of external states, [2] as well as on the genus g, through some coefficients $C_{g,N}$, we find by factorizing the N-point g–loop amplitude into an $N+1$-point g_1–loop amplitude times a 1-point g_2–loop amplitude times a propagator (where $g_1 + g_2 = g$), that

$$C_{g_1+g_2,N} \;\propto\; C_{g_1,N+1}\; C_{g_2,1} \tag{3.1}$$

with a proportionality constant independent of g_1, g_2 and N. Setting $g_1 = 0$ one gets

$$C_{g,N} \;\propto\; C_{0,N+1}\; C_{g,1} \;, \tag{3.2}$$

so that the dependence on N can be studied at tree level. Again by factorization, at tree level one gets

$$C_{0,N_1+N_2} \;\propto\; C_{0,N_1+1}\; C_{0,N_2+1} \;, \tag{3.3}$$

and if we put $N_2 = 2$ this implies that the ratio $C_{0,N+2}/C_{0,N+1}$ is independent of N or, in other words, that $C_{0,N} \propto (\mathcal{M})^N$ for some constant $\mathcal{M}$. So we may write

$$C_{g,N} \;=\; C_g\,(\mathcal{M})^N \;, \tag{3.4}$$

[2] In this section only we drop the label *tot* on N_{tot}.

and if we absorb a factor of $\mathcal{M}$ into the normalization of all vertex operators we are then left with an overall normalization constant C_g depending only on the genus.

To determine the value of C_g we adopt the method proposed in refs. [4,13]: To consider the elastic scattering of two gravitons in the Regge regime of very high center-of-mass energy and small energy transfer and impose that the leading part of the g-loop amplitude assumes the universal form needed for the eikonal resummation [14].

In order to get started we need the expression for the graviton vertex operator including the proper normalization which was found in refs. [15,4]:

$$\mathcal{V}^{(-1)}_{|\mathrm{grav}\rangle}(z,\bar{z}) = i\frac{\kappa}{\pi}\bar{\epsilon}\cdot\bar{\partial}X(\bar{z})\epsilon\cdot\psi(z)e^{-\phi(z)}e^{ik\cdot X(z,\bar{z})} \ , \tag{3.5}$$

where $k^2 = 0$ and we wrote the graviton polarization on the factorized form $\bar{\epsilon}\otimes\epsilon$ with $\epsilon\cdot k = \bar{\epsilon}\cdot k = 0$. Our conventions for the operator fields can be found in Appendix A. Like in eq. (2.4) we suppressed the cocycle factor which ensures that the superghost operator $e^{-\phi} = \delta(\gamma)$ anticommutes with all other fermions on the world-sheet.

By picture changing (3.5) we arrive at

$$\begin{aligned}\mathcal{V}^{(0)}_{|\mathrm{grav}\rangle}(z,\bar{z}) = & \lim_{w\to z}\Pi(w)\mathcal{V}^{(-1)}_{|\mathrm{grav}\rangle}(z,\bar{z}) \\ = & \frac{\kappa}{\pi}\bar{\epsilon}\cdot\bar{\partial}X(\bar{z})\left[\epsilon\cdot\partial X(z) - ik\cdot\psi(z)\epsilon\cdot\psi(z)\right]e^{ik\cdot X(z,\bar{z})} \ .\end{aligned} \tag{3.6}$$

The expressions for $\mathcal{V}^{(-1)}_{\langle\mathrm{grav}|}$ and $\mathcal{V}^{(0)}_{\langle\mathrm{grav}|}$ are identical to eqs. (3.5) and (3.6), as long as the polarizations $\epsilon, \bar{\epsilon}$ are taken to be real, and we ascribe to the outgoing graviton a momentum with $k^0 < 0$.

The calculation of the four-graviton g-loop amplitude in the Regge limit starting from eq. (2.2) is different from the one in ref. [4] which was performed using the manifestly world-sheet supersymmetric formulation of the heterotic string. In fact it is much harder, because even after changing the graviton vertex operators into the (0) picture there remains $2g-2$ PCOs at arbitrary points. To obtain the universal form of the amplitude in the pinching limit relevant for the Regge regime, where the world-sheet degenerates into a ladder-like configuration consisting of two "fast legs" connected by $g+1$ long tubes, one should insert $g-1$ PCOs on each of the two "fast legs". (Other choices are of course possible but will lead to the presence of total derivatives that make the leading behaviour of the amplitude rather obscure.) Even subject to this constraint there still remains $2g-2$ PCO insertion points, the dependence on which only drops out at the very end of the calculation.

In the end we recover the standard result [4] pertaining to $D=4$ space-time dimensions,

$$C_g = \left(\frac{2\kappa^2}{\alpha'}\right)^{g-1} \left(\frac{1}{2\pi}\right)^{5g-3} (\alpha')^{-2} \tag{3.7}$$

and the sign factor $(-1)^{g-1}$ explicitly displayed in eq. (2.2). The origin of this sign is not too hard to understand. It is needed to compensate the identical sign which appears when we disentangle the anticommuting superghost factors e^{ϕ} and the orbital supercurrents $T_F^{[X,\psi]}$ in the product of the $2g-2$ PCOs

$$\prod_{\alpha=1}^{2g-2} \left(e^{\phi(w_\alpha)} T_F^{[X,\psi]}(w_\alpha)\right) = (-1)^{g-1} \left(\prod_{\alpha=1}^{2g-2} e^{\phi(w_\alpha)}\right) \left(\prod_{\alpha=1}^{2g-2} T_F^{[X,\psi]}(w_\alpha)\right) . \tag{3.8}$$

The other three terms present in the PCO (2.4) do not contribute to the leading behaviour of the amplitude in the Regge regime.

A comment about the spin structure summation coefficient in eq. (2.2) might be in order at this point: We fix C_g by considering the four-graviton g-loop amplitude in the Regge regime. However, only the 2^g spin structures responsible for graviton exchange contribute to the leading, universal part of the amplitude. How do we know that the normalization we obtain is also correct for all the other spin structures? The answer to this has already been given in section 2: The requirement that the amplitude factorizes properly in the limit where all loops are taken far apart implies that the spin structure summation coefficient should be a product of one-loop summation coefficients. These are in turn specified by the requirement that the one-loop partition function should be modular invariant, once a (physically sensible) choice of GSO projection has been made [8,9]. [3]

4. The relation between $\mathcal{W}_{|\lambda\rangle}$ and $\mathcal{W}_{\langle\lambda|}$

We now consider in detail the connection between the vertex operators describing incoming and outgoing string states.

[3] Strictly speaking modular invariance of the one-loop partition function does not specify the summation coefficient for those spin structures where one (or more) of the free fermions on the world-sheet develop a zero mode, because these spin structures give zero contribution to the partition function. In order to check that no extra phase factors appear in these cases one may for example consider the factorization of a two-loop vacuum amplitude into one-loop tadpoles [9]. We carried out this check explicitly in the framework of Kawai-Lewellen-Tye [8] heterotic string models.

What we are looking for is the map which, given the vertex operator $\mathcal{W}_{|\lambda\rangle}$ describing an incoming string state, gives us the vertex operator $\mathcal{W}_{\langle\lambda|}$ describing the same string state but outgoing. As we saw in section 1 this map is just given by hermitean conjugation in the framework of quantum field theory. In string theory this cannot be the whole story, because if the operator field $\mathcal{W}_{|\lambda\rangle}$ creates the ket-state $|\lambda\rangle$ in the usual sense, $|\lambda\rangle = \lim_{\zeta,\bar{\zeta}\to 0} \mathcal{W}_{|\lambda\rangle}(\zeta,\bar{\zeta})|0\rangle$, then by definition the hermitean conjugate operator field creates the corresponding bra-state, $\langle\lambda| = \lim_{\zeta,\bar{\zeta}\to 0}\langle 0| \left(\mathcal{W}_{|\lambda\rangle}(\zeta,\bar{\zeta})\right)^\dagger$. But in eq. (2.2) both $\mathcal{V}_{\langle\lambda|}$ and $\mathcal{V}_{|\lambda\rangle}$ are vertex operators that create ket states when acting on the conformal ket vacuum.

So we need to compose two-dimensional hermitean conjugation with some other transformation which also maps a vertex operator creating ket-states into a vertex operator creating bra-states. This transformation should be a symmetry of any 2-dimensional conformal field theory on the sphere. The obvious choice is the *BPZ conjugation* [16] (see also [17]).

Therefore we now quickly review our conventions on hermitean conjugation and BPZ conjugation in conformal field theory. After that we will propose a map from $\mathcal{W}_{|\lambda\rangle}$ to $\mathcal{W}_{\langle\lambda|}$ which is just an unknown phase factor times the combination of BPZ and hermitean conjugation. In the next section we will check that our guess indeed gives the right map, and in the process the phase factor will be determined.

4.1 Two-dimensional hermitean conjugation

In this section we review our conventions on hermitean conjugation, see also refs. [5,12]. We define the hermitean conjugate of all elementary operators in the conformal field theory by specifying the hermitean conjugate of the corresponding oscillators, with the further understanding that hermitean conjugation also complex conjugates all complex numbers and inverts the order of the operators.

For example, if

$$\Phi_\Delta(z) = \sum_n \phi_n z^{-n-\Delta} \tag{4.1}$$

is a primary chiral conformal field of conformal dimension Δ, then the hermitean conjugate of this field is

$$(\Phi_\Delta(z))^\dagger \;=\; \left(\frac{1}{z^*}\right)^{2\Delta} \widehat{\Phi}_\Delta(\frac{1}{z^*}) \ , \tag{4.2}$$

where z^* denotes the complex conjugate of z (we think of z and $\bar{z}$ as independent complex variables, so that z^* and $\bar{z}$ need not be equal) and

$$\widehat{\Phi}_\Delta(z) = \sum_n \phi^\dagger_{-n} z^{-n-\Delta} \tag{4.3}$$

is a primary conformal field of the same dimension as Φ_Δ. We say that a field Φ_Δ is hermitean (anti-hermitean) when $\widehat{\Phi}_\Delta = +\Phi_\Delta$ $(-\Phi_\Delta)$.

The hermiticity properties are made more complicated by the presence of the reparametrization ghosts, because on the sphere the basic nonvanishing correlator is $\langle \bar{c}_{-1}\bar{c}_0\bar{c}_1 c_{-1}c_0c_1\rangle$ where (since $c^\dagger_n = c_{-n}$) the operator involved is explicitly anti-hermitean. Therefore either one has to postulate an imaginary value for this correlator or one has to relinquish the property $\langle M|A|N\rangle = +\langle N|A^\dagger|M\rangle^*$ of matrix elements involving ghost degrees of freedom. We prefer the second option. We define

$$\langle\ |c_{-1}c_0c_1|^2\ \rangle = \langle \bar{c}_{-1}\bar{c}_0\bar{c}_1 c_{-1}c_0c_1\rangle = +1 \tag{4.4}$$

and this implies that

$$\langle M|A|N\rangle = -\langle N|A^\dagger|M\rangle^* \tag{4.5}$$

in the presence of ghosts. As a special case of this

$$\langle M|c_0\bar{c}_0A|N\rangle = \langle N|c_0\bar{c}_0A^\dagger|M\rangle^* \tag{4.6}$$

for any operator A not involving the modes b_0 or $\bar{b}_0$.

A list of hermiticity properties for the fields relevant in four-dimensional heterotic string models constructed using free fermions can be found in Appendix B.

4.2 BPZ invariance in conformal field theories

Consider a conformal field theory on the cylinder. Introduce complex coordinates $z = \exp\{i(\sigma+\tau)\}$ and $\bar{z} = \exp\{i(-\sigma+\tau)\}$ and rotate to Euclidean time $\tau \to -i\tau$. Changing sign on τ and σ simultaneously gives rise to the Belavin-Polyakov-Zamolodchikov (BPZ) transformation $z \to 1/z$ [16,17]. This transformation defines a globally holomorphic diffeomorfism on the sphere.

At the level of the operator fields, the transformation changes the coordinate system from (z) to (w) where $w = 1/z$:

$$\Phi(z=\zeta) \xrightarrow{\text{BPZ}} \Phi(w=\zeta)\ . \tag{4.7}$$

For a primary conformal field of dimension Δ

$$\Phi_\Delta(w=\zeta) = e^{-i\epsilon\pi\Delta}\left(\frac{1}{\zeta}\right)^{2\Delta}\Phi_\Delta\left(z=\frac{1}{\zeta}\right) , \tag{4.8}$$

where for non-integer conformal dimensions we have to choose a specific phase for -1, [4] parametrized by an odd integer ϵ, when forming the transformation factors

$$\frac{dz}{dw} = e^{-i\epsilon\pi}\frac{1}{w^2} \qquad \text{and} \qquad \frac{dw}{dz} = e^{+i\epsilon\pi}\frac{1}{z^2} . \tag{4.9}$$

The BPZ transformation does not reverse the order of operators and it leaves all complex numbers unchanged. It cannot itself be generated by any operator acting on ket states. Instead it defines a map from ket-states to bra-states as follows:

$$|\Phi\rangle \;\equiv\; \lim_{\zeta\to 0}\Phi(z=\zeta)|0\rangle \;\;\stackrel{\text{BPZ}}{\longrightarrow}\;\; \langle\Phi^{\text{BPZ}}| \;\equiv\; \lim_{\zeta\to 0}\langle 0|\Phi(w=\zeta) . \tag{4.10}$$

The label "BPZ" on the state $\langle\Phi^{\text{BPZ}}|$ is necessary in order to avoid confusion with the bra state $\langle\Phi| \equiv \lim_{\zeta\to 0}\langle 0|\left(\Phi(z=\zeta)\right)^\dagger$ defined by hermitean conjugation, because this will in general differ from $\langle\Phi^{\text{BPZ}}|$. (Another possibility, preferred by many authors, is to take BPZ conjugation as the defining map from ket to bra and introduce instead a label $\langle\Phi^{\text{h.c.}}|$ on the state defined by hermitean conjugation.)

4.3 Composing BPZ and hermitean conjugation

The composition of BPZ and hermitean conjugation gives a map from ket to ket

$$|\Phi\rangle \;\longrightarrow\; \left(\langle\Phi^{\text{BPZ}}|\right)^\dagger \;=\; |\Phi^{\text{BPZ}}\rangle \tag{4.11}$$

which acts on the primary conformal fields as follows

$$\begin{aligned}\Phi_{\Delta,\bar\Delta}(z=\zeta,\bar z=\bar\zeta) \;\longrightarrow\; & \left(\Phi_{\Delta,\bar\Delta}(w=\zeta,\bar w=\bar\zeta)\right)^\dagger \\ =\; & e^{i\epsilon\pi(\Delta-\bar\Delta)}\widehat{\Phi}_{\Delta,\bar\Delta}(z=\zeta^*,\bar z=\bar\zeta^*) .\end{aligned} \tag{4.12}$$

Notice that for fields with non-integer value of $\Delta-\bar\Delta$, BPZ and hermitean conjugation do not commute. However, this is not a problem for vertex operators describing BRST-invariant on-shell string states, which satisfy $\Delta=\bar\Delta=0$.

[4] In their original paper [16], Belavin, Polyakov and Zamolodchikov avoided this problem by considering instead the conformal transformation $z\to -1/z$, but we prefer to consider $z\to 1/z$, in accordance with most subsequent authors.

The transformation (4.12) is our educated guess for the map taking $\mathcal{W}_{|\lambda\rangle}$ into $\mathcal{W}_{\langle\lambda|}$, only we will allow the possibility that some phase factor χ may appear. In other words, our ansatz is that if some incoming string state with definite quantum numbers is created, in the superghost charge q picture, by the vertex operator $\mathcal{W}^{(q)}_{|\lambda\rangle}$,

$$|\lambda\rangle \;=\; \lim_{z,\bar{z}\to 0} \mathcal{W}^{(q)}_{|\lambda\rangle}(z,\bar{z})|0\rangle \;, \tag{4.13}$$

then the vertex operator we have to use in the "master formula" (2.2) to obtain the T-matrix element involving the outgoing state $\langle\lambda|$ is given by

$$\mathcal{W}^{(q)}_{\langle\lambda|}(z=\zeta,\bar{z}=\bar{\zeta}) \;\equiv\; \chi_q \left(\mathcal{W}^{(q)}_{|\lambda\rangle}(w=\zeta^*,\bar{w}=\bar{\zeta}^*)\right)^\dagger \;. \tag{4.14}$$

As was emphasized at the beginning of section 4, the operator $\mathcal{W}^{(q)}_{\langle\lambda|}$, like any vertex operator, creates a state by acting on the ket vacuum. From the definitions (4.14) and (4.10) we find this state to be

$$\lim_{\zeta,\bar{\zeta}\to 0} \mathcal{W}^{(q)}_{\langle\lambda|}(z=\zeta,\bar{z}=\bar{\zeta})|0\rangle \;=\; \chi_q|\lambda^{\mathrm{BPZ}}\rangle \;. \tag{4.15}$$

In other words, we obtain the T-matrix element involving the bra-state $\langle\lambda|$ by inserting into the Polyakov path integral an operator creating the state $\chi_q|\lambda^{\mathrm{BPZ}}\rangle$. Notice that whereas the state $|\lambda\rangle$ always has $k^0>0$, the state $|\lambda^{\mathrm{BPZ}}\rangle$ has $k^0<0$.

Since the combination of BPZ and hermitean conjugation maps

$$L_0 \to L_0 \qquad \text{and} \qquad Q_{BRST} \to -Q_{BRST} \;, \tag{4.16}$$

and since BPZ conjugation is a world-sheet symmetry on the sphere, it follows that if the state $|\lambda\rangle$ is a physical on-shell state, $L_0|\lambda\rangle = Q_{BRST}|\lambda\rangle = 0$, then so is the state $\chi_q|\lambda^{\mathrm{BPZ}}\rangle$, regardless of what value we choose for the phase χ_q. It is less clear that the map (4.14) is also consistent with the GSO projection, i.e. that $|\lambda\rangle$ satisfies the GSO projection conditions if and only if $|\lambda^{\mathrm{BPZ}}\rangle$ does, because the two states will in general reside in different sectors of the string theory. An explicit proof in the framework of a Kawai-Lewellen-Tye (KLT) type heterotic string model is given in Appendix C.

If we restrict ourselves to BRST invariant on-shell string states, both $\mathcal{W}_{|\lambda\rangle}$ and $\widehat{\mathcal{W}}_{|\lambda\rangle}$ are primary conformal fields of dimension zero, and eq. (4.14) becomes

$$\mathcal{W}^{(q)}_{\langle\lambda|}(\zeta,\bar{\zeta}) = \chi_q \widehat{\mathcal{W}}^{(q)}_{|\lambda\rangle}(\zeta,\bar{\zeta}) \;. \tag{4.17}$$

We will now proceed to verify our ansatz (4.17) by considering the amplitude for the string state $|\lambda\rangle$ to emit (absorb) a very soft graviton. We will find that the phase factor χ_q, as anticipated by our notation, depends only on the choice of picture. In particular, if we restrict ourselves to the pictures $q = -1$ and $q = -1/2$, the phase factor χ_q depends only on whether the string state is a space-time boson or a space-time fermion. At the same time we will be able to determine the correct overall normalization of the vertex operators to be used in the formula (2.2) for the T-matrix element.

5. Normalization of vertex operators

In this section we consider the computation of the tree-amplitude for some given on-shell string state to absorb or emit a very soft graviton. We perform the analysis for a generic four dimensional heterotic string theory where the graviton vertex operator has the form of eq. (3.5), but the argument can be readily applied to other string models.

We first discuss the case of space-time bosonic states and then the case of the space-time fermionic states.

5.1 Normalization of space-time bosonic vertex operators

We first recall what is the situation in field theory. Consider a basis of propagating bosonic particle states with momentum p, labelled by an index N, in terms of which the propagator assumes the diagonal form $P_{MN}/(p^2+m_N^2)$ where $P_{MN} = +\delta_{M,N}$ for physical states and $P_{MN} = -\delta_{M,N}$ for possible negative norm states. For example, for a photon with space-time vector index $M = \mu$ we have $P_{MN} = \eta_{\mu\nu}$.

The tree-level T-matrix element for such a particle to emit (absorb) a graviton contains a universal term which, in the limit where the graviton momentum is zero, assumes the form

$$\begin{aligned} -2\kappa\,\epsilon\cdot p\;\bar\epsilon\cdot p\;P_{MN} &= -4\frac{\kappa}{\alpha'}\,\epsilon\cdot k\;\bar\epsilon\cdot k\;P_{MN} \\ &= -C_0\left(\frac{\kappa}{\pi}\right)^3\epsilon\cdot k\;\bar\epsilon\cdot k\;P_{MN}\;, \end{aligned} \tag{5.1}$$

where we wrote the graviton polarization on the factorized form $\epsilon\otimes\bar\epsilon$ and C_0 is the overall normalization constant for string tree amplitudes, given by eq. (3.7). The behaviour (5.1)

describes the canonical coupling of gravity to the $p_\mu p_\nu$-part of the energy-momentum tensor of the propagating particle.

The sign of the amplitude (5.1) obviously depends on the sign convention for the graviton field $h_{\mu\nu}$. Eq. (5.1) corresponds to the expansion

$$g_{\mu\nu} = \eta_{\mu\nu} - 2\kappa \left(h_{\mu\nu} + \lambda \eta_{\mu\nu} h^{\sigma}{}_{\sigma} \right) + \mathcal{O}(h^2) \tag{5.2}$$

regardless of the coefficient λ chosen for the trace term.

The sign chosen for the graviton vertex operator (3.5) is in agreement with this convention, as one may check by computing the 3-graviton tree amplitude from eqs. (3.5) and (2.2) and comparing with eq. (5.1) in the case where the state $|M\rangle$ is itself a graviton.

Consider now computing the universal part (5.1) of the graviton absorption amplitude at genus zero in string theory. We consider a complete set of space-time bosonic string states $|N, k\rangle$, labelled by N, built from the superghost vacuum $|q = -1\rangle$, satisfying $b_0 = \bar{b}_0 = 0$ and having definite momentum k. We may think of N as specifying physical quantities such as helicity, charges and family labels.

The T-matrix element for the process "$N +$ graviton $\rightarrow M$" is given by

$$T^0(M, k|\text{graviton}; N, k) \;=\; -C_0 \langle \mathcal{W}^{(-1)}_{\langle M,k|}(z_1, \bar{z}_1) \mathcal{W}^{(0)}_{|\text{grav}\rangle}(z, \bar{z}) \mathcal{W}^{(-1)}_{|N,k\rangle}(z_2, \bar{z}_2) \rangle \;, \tag{5.3}$$

where we have to use the graviton vertex operator in the superghost charge (0) picture, given by eq. (3.6), and the states $|N, k\rangle$ and $\langle M, k|$ are now assumed to be physical, so that $\mathcal{W}^{(-1)}_{|N,k\rangle}$ and $\mathcal{W}^{(-1)}_{\langle M,k|}$ are primary conformal fields of dimension zero.

By projective invariance on the sphere we can fix $z_1 = \infty$, $z = 1$ and $z_2 = 0$; and since the $\mathcal{W}^{(-1)}_{\langle M,k|}$ vertex operator is assumed to have conformal dimension zero we can evaluate it in the coordinate system (w), where $w = 1/z$, without introducing any transformation factor. In so doing we just undo the BPZ transformation in the definition eq. (4.14) of the operator $\mathcal{W}^{(-1)}_{\langle M,k|}$ and obtain

$$\langle 0|\mathcal{W}^{(-1)}_{\langle M,k|}(w = \bar{w} = 0) \;=\; \chi_{-1} \;\langle 0| \left(\mathcal{W}^{(-1)}_{|M,k\rangle}(z = \bar{z} = 0) \right)^{\dagger} \;=\; \chi_{-1} \;\langle M, k| \;. \tag{5.4}$$

Accordingly eq. (5.3) becomes

$$T^0(M, k|\text{graviton}; N, k) \;=\; -\chi_{-1} \; C_0 \left(\frac{\kappa}{\pi} \right) \langle M, k| c(1) \bar{c}(1) \bar{\epsilon} \cdot \bar{\partial} X(1) \, \epsilon \cdot \partial X(1) |N, k\rangle \;. \tag{5.5}$$

Here we may expand the fields c, $\bar{c}$, ∂X and $\bar{\partial} X$ in oscillators. Only modes with $L_0 = \bar{L}_0 = 0$ can contribute to the "universal" part (5.1) of the amplitude. This is because this part

of the amplitude, like that of a freely propagating string state, conserves $L_0(X^\mu)$, $\bar{L}_0(X^\mu)$, $L_0(b,c)$ and $\bar{L}_0(\bar{b},\bar{c})$. We may imagine the basis $|N,k\rangle$ of string states to diagonalize all these operators. Then for $n \neq 0$ we may write e.g.

$$\alpha_n^\mu = -\frac{1}{n}\left[L_0(X^\mu), \alpha_n^\mu\right] \tag{5.6}$$

and this vanishes between the states $\langle M,k|$ and $|N,k\rangle$ since by assumption they have the same value of $L_0(X^\mu)$.

We are thus left with

$$T^0(M,k|\text{graviton};N,k) = -\chi_{-1}\ C_0\left(\frac{\kappa}{\pi}\right)\ \epsilon\cdot k\ \bar{\epsilon}\cdot k\ \langle M,k|\bar{c}_0 c_0|N,k\rangle + \dots\ , \tag{5.7}$$

where "$+\dots$" denotes possible other terms in the amplitude with a different kinematical structure than the universal part (5.1). By eq. (4.6) the matrix $\langle M,k|\bar{c}_0 c_0|N,k\rangle$ is manifestly hermitean and by an appropriate choice of basis it may be diagonalized such that

$$\langle M,k|\bar{c}_0 c_0|N,k\rangle = \left(\mathcal{N}^{\text{bos}}_{|M,k\rangle}\right)^* \mathcal{N}^{\text{bos}}_{|N,k\rangle}\ P_{MN}\ , \tag{5.8}$$

where either $P_{MN} = 0$ (so that the state does not propagate) or $|P_{MN}| = \delta_{M,N}$. Our conventions (4.4) imply that $P_{MN} = +\delta_{M,N}$ for all physical external states but $-\delta_{M,N}$ for negative norm states (such as the "timelike" photon). The factor $\mathcal{N}^{\text{bos}}_{|N,k\rangle}$ specifies the overall normalization of the state $|N,k\rangle$. By inserting eq. (5.8) into eq. (5.7) we obtain finally the correct result (5.1) *if* we take the phase factor introduced in eq. (4.14) to be $\chi_{-1} = 1$ and choose the normalization constant to be the same for all states, $\mathcal{N}^{\text{bos}}_{|M,k\rangle} = \mathcal{N}^{\text{bos}}_{|N,k\rangle}$, given by

$$\left|\mathcal{N}^{\text{bos}}_{|N,k\rangle}\right| = \frac{\kappa}{\pi}\ . \tag{5.9}$$

In summary,

$$\mathcal{W}^{(-1)}_{\langle N,k|}(z,\bar{z}) = +\widehat{\mathcal{W}}^{(-1)}_{|N,k\rangle}(z,\bar{z}) \qquad \text{for physical spacetime bosons}\ , \tag{5.10}$$

and the proper normalization of the state $|N,k\rangle$ is given by

$$\langle M,k|\bar{c}_0 c_0|N,k\rangle = \left(\frac{\kappa}{\pi}\right)^2 P_{MN}\ . \tag{5.11}$$

Since by definition $|N,k\rangle = \lim_{\zeta,\bar{\zeta}\to 0}\mathcal{W}_{|N,k\rangle}(\zeta,\bar{\zeta})|0\rangle$, eq. (5.11) specifies the normalization of the vertex operator up to a complex phase factor. If the vertex operator $\mathcal{W}^{(-1)}_{|N,k\rangle}$ is

complex, i.e. not proportional to $\widehat{\mathcal{W}}^{(-1)}_{|N,-k\rangle}$, [5] there is probably no fundamental reason to prefer any specific value of the overall complex phase factor, just as in field theory the phase of a complex field is an unphysical degree of freedom. If, on the other hand, $\mathcal{W}^{(-1)}_{|N,k\rangle}$ *is* proportional to $\widehat{\mathcal{W}}^{(-1)}_{|N,-k\rangle}$ it becomes natural to impose a reality condition, which we can take to be

$$\mathcal{W}^{(-1)}_{|N,k\rangle} = +\widehat{\mathcal{W}}^{(-1)}_{|N,-k\rangle} \qquad \text{or} \qquad \mathcal{V}^{(-1)}_{|N,k\rangle} = -\widehat{\mathcal{V}}^{(-1)}_{|N,-k\rangle} \tag{5.12}$$

in agreement with the choice made for the graviton vertex operator (3.5). This implies that $\mathcal{W}^{(-1)}_{\langle N,k|} = \mathcal{W}^{(-1)}_{|N,-k\rangle}$. Even in this case there remains a choice a sign for the vertex operator. This is completely dependent on convention, just like the sign of the graviton field in the expansion (5.2).

5.2 Normalization of space-time fermionic vertex operators

We now consider the case of space-time fermions. The field theory description is now more complicated than in the case of space-time bosons, since the graviton field should be described in terms of the vierbein, e^{μ}_{m}. The canonical coupling to gravity of a Dirac fermion, labelled by an index N, is given by the action

$$\int \mathrm{d}^4x \; e \; \overline{\psi}_M \left\{\gamma^m e^{\mu}_{m} \partial_\mu + m\right\} \psi_N \; P^{MN} \ , \tag{5.13}$$

where we ignore the spin-connection terms which all involve derivatives of the vierbein and thus give rise to terms in the fermion-fermion-graviton amplitude proportional to the graviton momentum. When expanding e^{μ}_{m} around the flat background we can ignore the deviation of $e = \det\{e^{\mu}_{m}\}$ from unity since this gives rise only to terms proportional to the trace of the graviton field. One obtains the following expression, analogous to eq. (5.1) for the universal part of the fermion-fermion-graviton T-matrix element at tree level:

$$-i\kappa \overline{u}(\vec{p},\eta)\gamma^\nu p^\mu u(\vec{p},\eta)\epsilon_\nu \bar{\epsilon}_\mu \; P_{MN} \ , \tag{5.14}$$

where, by virtue of the Gordon identity

$$\overline{u}(\vec{p},\eta)\gamma^\nu u(\vec{p},\eta') = -2ip^\nu \delta_{\eta,\eta'} \ , \tag{5.15}$$

we recover the bosonic result (5.1), as dictated by the principle of equivalence.

[5] Notice that if $\mathcal{W}^{(-1)}_{|N,k\rangle}$ is proportional to $\exp(ik\cdot X)$ then $\widehat{\mathcal{W}}^{(-1)}_{|N,k\rangle}$ is proportional to $\exp(-ik\cdot X)$.

In the string theory analysis we again consider a complete set of states $|N,k\rangle$, labelled by N, now built from the superghost vacuum $|q=-1/2\rangle$, again satisfying $b_0 = \bar{b}_0 = 0$ and having a definite momentum k.

We may now proceed exactly as in section 5.1, only now we have to use the superghost charge (-1) version of the graviton vertex operator, given by eq. (3.5). In the limit of vanishing graviton momentum we obtain

$$\begin{aligned} T^0(M,k|\text{graviton};N,k) &= -C_0 \langle \mathcal{W}^{(-1/2)}_{\langle M,k|}(z_1,\bar{z}_1)\mathcal{W}^{(-1)}_{|\text{grav}\rangle}(z,\bar{z})\mathcal{W}^{(-1/2)}_{|N,k\rangle}(z_2,\bar{z}_2)\rangle \\ &= -\chi_{-1/2}\, C_0 \,\langle M,k|\mathcal{W}^{(-1)}_{|\text{grav}\rangle}(1)|N,k\rangle \ . \end{aligned} \tag{5.16}$$

As in the bosonic case only zero-mode operators contribute to the part of the amplitude in which we are interested, so that

$$T^0(M,k|\text{graviton};N,k) \;=\; \chi_{-1/2}\, C_0 \frac{\kappa}{\pi}\, \bar{\epsilon}\cdot k\ \epsilon_\nu \langle M,k|\bar{c}_0 c_0 \psi^\nu_0 \delta(\gamma_0)|N,k\rangle + \ldots \ . \tag{5.17}$$

Here we may recognize the form (5.14) of the result obtained in field theory, since the zero mode ψ^ν_0 of the operator field ψ^ν furnishes a representation of the Clifford algebra, and so is completely analogous to the gamma matrix γ^ν appearing in the expression (5.14).

The matrix $\langle M,k|\bar{c}_0 c_0 \psi^\nu_0 \delta(\gamma_0)|N,k\rangle$ transforms as a space-time vector and therefore has to be proportional to the momentum k^ν. Since ψ^ν_0 and $\delta(\gamma_0)$ anti-commute it is manifestly anti-hermitean (q.v. eq. (4.6)) and by choosing an appropriate basis it can be diagonalized such that

$$\langle M,k|\bar{c}_0 c_0 \psi^\nu_0 \delta(\gamma_0)|N,k\rangle = iYk^\nu \left(\mathcal{N}^{\text{ferm}}_{|M,k\rangle}\right)^* \mathcal{N}^{\text{ferm}}_{|N,k\rangle}\ P_{MN} \ . \tag{5.18}$$

In section 7 we will explicitly derive this formula in the context of a KLT heterotic string model. It is quite analogous to eq. (5.8). The factor of i reflects the fact that the matrix on the left-hand side is anti-hermitean and (as we shall see in section 7) the constant factor $Y = \pm 1$ depends on the conventions chosen for the spin fields. Finally, $P_{MN} = +\delta_{M,N}$ for physical states, as always. Like in the bosonic case the factor $\mathcal{N}^{\text{ferm}}_{|N,k\rangle}$ specifies the normalization of the string state $|N,k\rangle$. If we insert eq. (5.18) into eq. (5.17) we finally obtain

$$\begin{aligned} &T^0(M,k|\text{graviton};N,k) \\ &\quad= Y\ i\ \chi_{-1/2}\ C_0 \left(\mathcal{N}^{\text{ferm}}_{|M,k\rangle}\right)^* \mathcal{N}^{\text{ferm}}_{|N,k\rangle} \left(\frac{\kappa}{\pi}\right) \bar{\epsilon}\cdot k\ \epsilon\cdot k\ P_{MN} + \ldots \ , \end{aligned} \tag{5.19}$$

which reproduces the right result (5.1) assuming we choose

$$\chi_{-1/2} = iY \qquad \text{for spacetime fermions} \tag{5.20}$$

and fix the normalization of the states in the same universal way as for the bosons, $\mathcal{N}^{\text{ferm}}_{|M,k\rangle} = \mathcal{N}^{\text{ferm}}_{|N,k\rangle}$, and

$$\left|\mathcal{N}^{\text{ferm}}_{|N,k\rangle}\right| = \left|\mathcal{N}^{\text{bos}}_{|N,k\rangle}\right| = \frac{\kappa}{\pi} \ . \tag{5.21}$$

In summary

$$\mathcal{W}^{(-1/2)}_{\langle N,k|}(z,\bar{z}) \ = \ (iY)\,\widehat{\mathcal{W}}^{(-1/2)}_{|N,k\rangle}(z,\bar{z}) \qquad \text{for physical spacetime fermions,} \tag{5.22}$$

and the proper normalization of the string state is given by

$$\langle M,k|\bar{c}_0 c_0 \psi_0^\nu \delta(\gamma_0)|N,k\rangle = iY k^\nu \left(\frac{\kappa}{\pi}\right)^2 \ P_{MN} \ . \tag{5.23}$$

Since the PCO (2.4) is an anti-hermitean operator which satisfies Bose statistics, eqs. (5.10) and (5.22) can be generalized to the superghost charge q picture as follows

$$\mathcal{W}^{(q)}_{\langle N,k|}(z,\bar{z}) \ = \ \chi_q \ \widehat{\mathcal{W}}^{(q)}_{|N,k\rangle}(z,\bar{z}) \ = \ (-1)^{q+1} \ \widehat{\mathcal{W}}^{(q)}_{|N,k\rangle}(z,\bar{z}) \ . \tag{5.24}$$

For pictures of half-integer q (i.e. pictures describing space-time fermions) the phase factor $(-1)^{q+1}$ involves a choice of sign, which is parametrized by Y according to eq. (5.20), i.e. $(-1)^{1/2} = iY$.

6. Space-Time hermiticity

An important check on the correctness of our expressions (5.10) and (5.22) for $\mathcal{W}^{(q)}_{\langle N,k|}$ is provided by the requirement that the T-matrix element obtained from eq. (2.2) has the right hermiticity properties.

Unitarity requires that the tree-level T-matrix element is real except when the momentum flowing in some intermediate channel happens to be on the mass-shell corresponding to some physical state in the theory. In field theory the imaginary part appears as a result of the $i\epsilon$-prescription present in the propagator that happens to be on-shell. In string theory it appears as a result of some divergency in the integral over the Koba-Nielsen

(KN) variables that has to be treated in a way consistent with the $i\epsilon$-prescription in field theory [18,19,20].

What we can rather easily show is that as long as the integrals over the KN variables are convergent the expressions (5.10) and (5.22) lead to a hermitean T-matrix at tree level.

At genus zero the formula (2.2) can be rewritten as

$$\begin{aligned} & T^0(\lambda_1;\ldots;\lambda_{N_{out}}|\lambda_{N_{out}+1};\ldots;\lambda_{N_{out}+N_{in}}) \;= \\ & \quad - C_0 \int \left(\prod_{i=4}^{N_{tot}} \mathrm{d}^2 z_i\right) \; \langle \bar c(\bar z_1)\bar c(\bar z_2)\bar c(\bar z_3) c(z_1)c(z_2)c(z_3) \; \times \\ & \quad \left(\prod_{A=1}^{N_B+N_{FP}-2} \Pi(w_A)\right) \mathcal{V}_{\langle\lambda_1|}(z_1,\bar z_1)\ldots\mathcal{V}_{|\lambda_{N_{\rm tot}}\rangle}(z_{N_{\rm tot}},\bar z_{N_{\rm tot}})\rangle \; . \end{aligned} \tag{6.1}$$

The T-matrix is hermitean if and only if the quantity (6.1) equals

$$\begin{aligned} & \left[T^0(\lambda_{N_{out}+N_{in}};\ldots;\lambda_{N_{out}+1}|\lambda_{N_{out}};\ldots;\lambda_1)\right]^* \;= \\ & \quad + C_0 \int \left(\prod_{i=4}^{N_{\rm tot}} \mathrm{d}^2 z_i^*\right) \; \langle \left(\mathcal{V}_{|\lambda_1\rangle}(z_1,\bar z_1)\right)^\dagger \ldots \left(\mathcal{V}_{\langle\lambda_{N_{\rm tot}}|}(z_{N_{\rm tot}},\bar z_{N_{\rm tot}})\right)^\dagger \; \times \\ & \quad \left(\Pi(w_{N_B+N_{FP}-2})\right)^\dagger \ldots \left(\Pi(w_1)\right)^\dagger \; \left(c(z_3)\right)^\dagger \ldots \left(\bar c(\bar z_1)\right)^\dagger\rangle \; , \end{aligned} \tag{6.2}$$

where we used eq. (4.5).

In terms of the vertex operators $\mathcal{V}$ (where the $c\bar c$ factor present in $\mathcal{W}$ has been removed, q.v. eq. (2.3)) the relations (5.10) and (5.22) acquire an extra minus sign (because $c\bar c$ is an anti-hermitean operator):

$$\begin{aligned} \mathcal{V}^{(-1)}_{\langle\lambda|}(z,\bar z) &= -\widehat{\mathcal{V}}^{(-1)}_{|\lambda\rangle}(z,\bar z) \\ \mathcal{V}^{(-1/2)}_{\langle\lambda|}(z,\bar z) &= -iY\; \widehat{\mathcal{V}}^{(-1/2)}_{|\lambda\rangle}(z,\bar z) \; , \end{aligned} \tag{6.3}$$

which, by taking the hermitean conjugate, leads to the inverse relations

$$\begin{aligned} \mathcal{V}^{(-1)}_{|\lambda\rangle}(z,\bar z) &= -\widehat{\mathcal{V}}^{(-1)}_{\langle\lambda|}(z,\bar z) \\ \mathcal{V}^{(-1/2)}_{|\lambda\rangle}(z,\bar z) &= -iY\; \widehat{\mathcal{V}}^{(-1/2)}_{\langle\lambda|}(z,\bar z) \; . \end{aligned} \tag{6.4}$$

Since the operators $\mathcal{V}_{|\lambda\rangle}$ and $\mathcal{V}_{\langle\lambda|}$ have conformal dimensions $\Delta = \bar\Delta = 1$ we find for $i = 1,\ldots,N_{out}$:

$$\begin{aligned} \left(\mathcal{V}_{|\lambda_i\rangle}(z_i,\bar z_i)\right)^\dagger \;&=\; \left(\frac{1}{z_i^*}\frac{1}{\bar z_i^*}\right)^2 \widehat{\mathcal{V}}_{|\lambda_i\rangle}\left(\frac{1}{z_i^*},\frac{1}{\bar z_i^*}\right) \\ &= (\text{phase factor}) \times \left(\frac{1}{z_i^*}\frac{1}{\bar z_i^*}\right)^2 \times \mathcal{V}_{\langle\lambda_i|}\left(\frac{1}{z_i^*},\frac{1}{\bar z_i^*}\right) \; , \end{aligned} \tag{6.5}$$

where the phase factor we pick up is minus one for space-time bosons and iY for space-time fermions. By eqs. (6.4) we pick up exactly the same phase factor from vertex operators of the type $\mathcal{V}_{\langle\lambda|}$. This amounts to an overall sign $(-1)^{N_B+N_{FP}}$, N_{FP} being the number of space-time fermion pairs and N_B the number of space-time bosons. This sign exactly cancels the sign produced by the $N_B + N_{FP} - 2$ PCOs, which are anti-hermitean. Finally, reordering the ghost factors in (6.2) in accordance with eq. (6.1), we obtain a minus sign cancelling the one that was introduced by using eq. (4.5).

Since the transformation factors $(z_i^*)^{-2}(\bar{z}_i^*)^{-2}$ appearing in eq. (6.5) either cancels a similar one coming from the ghost operators (for $i = 1, 2, 3$), or is just the required jacobian to transform $\mathrm{d}^2 z_i$ into $\mathrm{d}^2\zeta_i$ where $\zeta_i = 1/z_i^*$ $(i \geq 4)$, we finally recover eq. (6.1) multiplied by a phase factor that, at the end, is just plus one. This concludes the proof that our relation between $\mathcal{W}^{(q)}_{\langle\lambda|}$ and $\mathcal{W}^{(q)}_{|\lambda\rangle}$ leads to a hermitean T-matrix at tree level away from the resonances.

7. An explicit example

In this section we provide an explicit example of the map (5.24) in the context of four-dimensional heterotic string models of the Kawai-Lewellen-Tye (KLT) type [8,9], where the internal degrees of freedom are described by 22 left-moving and 9 right-moving free complex fermions. We bosonize all these fermions (as well as the four Majorana fermions ψ^μ), using the explicit prescription for bosonization in Minkowski space-time proposed in ref. [12].

In this formulation any state of the conformal field theory (excluding the reparametrization ghosts) can be obtained by means of non-zero mode creation operators from the generic ground state which is specified by the space-time momentum k, the "momentum" $J_0^{(L)} = \mathbf{A}_L$ of the 33 bosons $\Phi_{(L)}$ introduced by the bosonization, and the superghost charge $J_0^{(34)} = q = \mathbf{A}_{34}$ which is (minus) the "momentum" of the field $\phi \equiv \Phi_{(34)}$ that is introduced when "bosonizing" the superghosts. Since $[J_0^{(L)}, \Phi_{(K)}] = \delta_K^{\,L}$, the operator creating such a ground state from the conformal vacuum is

$$S_{\mathbf{A}}(z,\bar{z})\; e^{ik\cdot X(z,\bar{z})} \;, \tag{7.1}$$

where

$$S_{\mathbf{A}}(z,\bar{z}) \equiv \prod_{L=1}^{34} e^{\mathbf{A}_L \Phi_{(L)}(z,\bar{z})} \left(C_{(L)}\right)^{\mathbf{A}_L} \;, \tag{7.2}$$

is a spin field operator and $C_{(L)}$ is a cocycle factor, see ref. [12] for details.

The range of values allowed for the $\mathbf{A}_L$ depends on the details of the KLT model we happen to consider, see refs. [8,11]. We assume the level-matching condition $L_0 - \bar{L}_0 = 0$ to be satisfied.

The hermitean conjugate of the operator $S_{\mathbf{A}}(z,\bar{z})$ can be computed using the hermiticity properties of the various fields, as outlined in Appendix B (see also ref. [12]). One finds

$$\widehat{S}_{\mathbf{A}}(z,\bar{z}) = \left(\sigma_1^{(33)}\mathbf{C}^{-1}\right)_{\mathbf{A}}{}^{\mathbf{B}} \; S_{\mathbf{B}}(z,\bar{z}) \; , \tag{7.3}$$

where

$$(\sigma_1^{(33)})_{\mathbf{AB}} = \left(\prod_{L=1}^{32} \delta_{\mathbf{A}_L,\mathbf{B}_L}\right) \; \delta_{\mathbf{A}_{33}+\mathbf{B}_{33},0} \; \delta_{\mathbf{A}_{34},\mathbf{B}_{34}} \tag{7.4}$$

and $\mathbf{C}^{-1}$ is the inverse of the "charge conjugation matrix"

$$\mathbf{C}_{\mathbf{AB}} = \left(\prod_{L=1}^{33} \delta_{\mathbf{A}_L+\mathbf{B}_L,0}\right) \; \delta_{\mathbf{A}_{34},\mathbf{B}_{34}} \; e^{i\pi \mathbf{A}\cdot Y\cdot \mathbf{B}} \tag{7.5}$$

defined in terms of the 34×34 cocycle matrix Y_{KL} (see refs. [12,11]).

The example we want to study is that of a physical space-time fermion described by a ground state. To obtain a BRST-invariant state one has to consider a vertex operator which involves a linear combination of spin fields,

$$\mathcal{V}^{(-1/2)}_{|\mathbf{V},k\rangle}(z,\bar{z}) \;=\; \frac{\kappa}{\pi}\, \mathbf{V}^{\mathbf{A}}_{(-\frac{1}{2})}(k)\, S_{\mathbf{A}}(z,\bar{z})\; e^{ik\cdot X(z,\bar{z})} \; , \tag{7.6}$$

where the spinor $\mathbf{V}^{\mathbf{A}}_{(-\frac{1}{2})}(k)$ has superghost charge $-1/2$, i.e. is proportional to $\delta_{\mathbf{A}_{34},-1/2}$, and satisfies a Dirac equation which can be obtained from the requirement that the 3/2-order pole in the operator product expansion (OPE) of the supercurrent $T_F^{[X,\psi]}$ with the operator (7.6) vanishes. If we define the gamma matrices by the OPE

$$\psi^\mu(z)S_{\mathbf{A}}(w,\bar{w}) \overset{\text{OPE}}{=} \frac{1}{\sqrt{2}}\left(\boldsymbol{\Gamma}^\mu\right)_{\mathbf{A}}{}^{\mathbf{B}}\, S_{\mathbf{B}}(w,\bar{w})\frac{1}{\sqrt{z-w}} + \ldots \; , \tag{7.7}$$

the Dirac equation assumes the matrix form

$$(\mathbf{V}_{(-\frac{1}{2})}(k))^T\; \mathbf{D}\,(k) = 0 \qquad \text{or} \qquad (\mathbf{D}\,(k))^T\, \mathbf{V}_{(-\frac{1}{2})}(k) = 0 \; , \tag{7.8}$$

where the Dirac operator is

$$\mathbf{D}\,(k) = k_\mu \boldsymbol{\Gamma}^\mu - \mathbf{M} \; , \tag{7.9}$$

$\mathbf{M}$ being a mass operator that we do not need to write down explicitly.

When the vertex operator is written as in eq. (7.6) we are no longer free to choose the normalization of the spinor $\mathbf{V}_{(-\frac{1}{2})}(k)$. It should be fixed in accordance with eq. (5.23). In the next subsection we will explicitly verify that the correct normalization is

$$(\mathbf{V}_{(-\frac{1}{2})}(k))^{\dagger}\ \mathbf{V}_{(-\frac{1}{2})}(k) = \sqrt{2}\ |k^0|\ , \tag{7.10}$$

which is analogous in structure to eq. (1.5).

By using eq. (7.3) in the expression (6.3) we find the "outgoing" vertex operator corresponding to (7.6) to be

$$\mathcal{V}^{(-1/2)}_{\langle\mathbf{V},k|}(z,\bar{z}) \;=\; -\chi_{-1/2}\frac{\kappa}{\pi}\left(\mathbf{V}^{\mathbf{A}}_{(-\frac{1}{2})}(k)\right)^{*}\left(\sigma_1^{(33)}\mathbf{C}^{-1}\right)_{\mathbf{A}}^{\ \ \mathbf{B}} S_{\mathbf{B}}(z,\bar{z})\ e^{-ik\cdot X(z,\bar{z})}\ , \tag{7.11}$$

where $\chi_{-1/2} = iY$.

7.1 A Sample Computation.

We will now explicitly compute the amplitude for a space-time fermion described by the vertex operator (7.6) to absorb a zero-momentum graviton. In particular we will obtain the relation (5.19) and show how the sign Y appearing in this formula is related to the choice of cocycles.

Inserting eqs. (7.6), (7.11) and (3.5) into eq. (5.16) we obtain:

$$\begin{aligned} T^0(\mathbf{V},k|\mathrm{graviton};\mathbf{V},k) =&\ -C_0\ \langle \mathcal{W}^{(-1/2)}_{\langle\mathbf{V},k|}(z_1,\bar{z}_1)\ \mathcal{W}^{(-1)}_{|\mathrm{grav}\rangle}(z,\bar{z})\mathcal{W}^{(-1/2)}_{|\mathbf{V},k\rangle}(z_2,\bar{z}_2)\rangle \\ =&\ i\chi_{-1/2}C_0\left(\frac{\kappa}{\pi}\right)^3\left(\mathbf{V}^{\mathbf{A}}_{(-\frac{1}{2})}(k)\right)^{*}\left(\sigma_1^{(33)}\mathbf{C}^{-1}\right)_{\mathbf{A}}^{\ \ \mathbf{B}}\mathbf{V}^{\mathbf{C}}_{(-\frac{1}{2})}(k)\ \epsilon_\mu\ \times \\ &\ \langle S_{\mathbf{B}}(z_1,\bar{z}_1)\psi^\mu(z)e^{-\Phi_{(34)}(z)}(C_{(34)})^{-1}S_{\mathbf{C}}(z_2,\bar{z}_2)\rangle\ \times \\ &\ \langle\bar{\epsilon}\cdot\bar{\partial}X(\bar{z})e^{-ik\cdot X(z_1,\bar{z}_1)}e^{ik\cdot X(z_2,\bar{z}_2)}\rangle\ \langle\bar{c}(\bar{z}_1)\bar{c}(\bar{z})\bar{c}(\bar{z}_2)c(z_1)c(z)c(z_2)\rangle\ . \end{aligned} \tag{7.12}$$

By explicit computation one finds

$$\begin{aligned} &\langle e^{-\Phi_{(34)}(z)}(C_{(34)})^{-1}\psi^\mu(z)S_{\mathbf{B}}(z_1,\bar{z}_1)S_{\mathbf{C}}(z_2,\bar{z}_2)\rangle\ = \\ &\frac{1}{\sqrt{2}}\left(\boldsymbol{\Gamma}^\mu\ \mathbf{C}_{(-1)}\right)_{\mathbf{B}\,\mathbf{C}}\frac{z_1-z_2}{(z-z_1)(z-z_2)}|z_1-z_2|^{-2(2+m^2)}\ , \end{aligned} \tag{7.13}$$

where m is the mass of the space-time fermion, $k^2+m^2=0$, and we introduced another family of "charge conjugation matrices" by

$$\left(\mathbf{C}_{(q)}\right)_{\mathbf{A}\mathbf{B}} = \left(\prod_{L=1}^{33}\delta_{\mathbf{A}_L+\mathbf{B}_L,0}\right)\ \delta_{\mathbf{A}_{34}+\mathbf{B}_{34}+q+2,0}\ e^{i\pi\mathbf{A}\cdot Y\cdot\mathbf{B}} \tag{7.14}$$

for any value of $q \in \mathbf{Z}$.

Similarly one finds

$$\begin{aligned} \langle \bar{\epsilon} \cdot \bar{\partial} X(\bar{z}) e^{-ik\cdot X(z_1,\bar{z}_1)} e^{ik\cdot X(z_2,\bar{z}_2)} \rangle &= i\bar{\epsilon} \cdot k \frac{\bar{z}_1 - \bar{z}_2}{(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2)} |z_1 - z_2|^{-2k^2} \\ \langle \, |c(z_1)c(z)c(z_2)|^2 \, \rangle &= |(z_1 - z)(z - z_2)(z_1 - z_2)|^2 \; . \end{aligned} \tag{7.15}$$

Substituting (7.13) and (7.15) into eq. (7.12) we obtain

$$\begin{aligned} & T^0(\mathbf{V}, k|\text{graviton}; \mathbf{V}, k) \; = \\ & \chi_{-1/2} C_0 \left(\frac{\kappa}{\pi}\right)^3 \frac{1}{\sqrt{2}} \; \bar{\epsilon} \cdot k \; \epsilon_\mu \left(\left(\mathbf{V}_{(-\frac{1}{2})}(k) \right)^\dagger \sigma_1^{(33)} \mathbf{C}^{-1} \boldsymbol{\Gamma}^\mu \mathbf{C}_{(-1)} \mathbf{V}_{(-\frac{1}{2})}(k) \right) \; . \end{aligned} \tag{7.16}$$

One may show that

$$\left(\boldsymbol{\Gamma}^\mu \mathbf{C}_{(-1)} \right)_{\mathbf{AB}} \; = \; (-1)^{\mathbf{A}_{34}+1/2} \left(\mathbf{C}_{(-1)} \boldsymbol{\Sigma} \left(\boldsymbol{\Gamma}^\mu \right)^T \right)_{\mathbf{AB}} \; , \tag{7.17}$$

where

$$\boldsymbol{\Sigma}_{\mathbf{AB}} \equiv \left(\prod_{L=1}^{34} \delta_{\mathbf{A}_L, \mathbf{B}_L} \right) \; \exp \left\{ i\pi \sum_{L=1}^{33} Y_{34,L} \mathbf{B}_L \right\} \tag{7.18}$$

and the sign $(-1)^{\mathbf{A}_{34}+1/2}$ is effectively equal to one, since the matrices appearing in eq. (7.16) are sandwiched between spinors with superghost charge $-1/2$. For the same reason the inverse charge conjugation matrix $\mathbf{C}^{-1}$ is effectively equal to $(\mathbf{C}_{(-1)})^{-1}$. Finally it is straightforward to verify that $\boldsymbol{\Gamma}^0$, as defined by eq. (7.7), may also be written on the form

$$\boldsymbol{\Gamma}^0 = iY_{34,33} \boldsymbol{\Sigma} \sigma_1^{(33)} \tag{7.19}$$

and since $\boldsymbol{\Sigma}$ and $\sigma_1^{(33)}$ anticommute, $(\boldsymbol{\Gamma}^0)^T = -\boldsymbol{\Gamma}^0$.

Inserting eqs. (7.17) and (7.19) into eq. (7.16) we obtain

$$\begin{aligned} & T^0(\mathbf{V}, k|\text{graviton}; \mathbf{V}, k) \; = \\ & - i\chi_{-1/2} Y_{34,33} \; C_0 \left(\frac{\kappa}{\pi}\right)^3 \frac{1}{\sqrt{2}} \; \bar{\epsilon} \cdot k \; \epsilon_\mu \left(\left(\mathbf{V}_{(-\frac{1}{2})}(k) \right)^\dagger (\boldsymbol{\Gamma}^0)^T (\boldsymbol{\Gamma}^\mu)^T \mathbf{V}_{(-\frac{1}{2})}(k) \right) \; . \end{aligned} \tag{7.20}$$

At this point we may use the Gordon-like identity

$$\left(\mathbf{V}_{(-\frac{1}{2})}(k) \right)^\dagger (\boldsymbol{\Gamma}^0)^T (\boldsymbol{\Gamma}^\mu)^T \; \mathbf{V}_{(-\frac{1}{2})}(k) = -\sqrt{2} k^\mu \; . \tag{7.21}$$

This equation can be proven directly using the Dirac equation (7.8), but it is easier to note that Lorentz covariance forces the right-hand side to be proportional to k^μ and then fix

the proportionality constant by setting $\mu = 0$ and using equation (7.10). Thus we finally obtain

$$T^0(\mathbf{V}, k|\text{graviton}; \mathbf{V}, k) \;=\; i\chi_{-1/2} Y_{34,33}\; C_0 \left(\frac{\kappa}{\pi}\right)^3 \bar{\epsilon}\cdot k\;\epsilon\cdot k\;. \tag{7.22}$$

This agrees with the correct result (5.1) provided we choose

$$\chi_{-1/2} = iY = iY_{34,33} \tag{7.23}$$

and shows that the sign Y appearing in eq. (5.18) should be identified with the component $Y_{34,33}$ of the cocycle matrix. At the same time we have verified the correctness of the normalization (7.10) for the spinor $\mathbf{V}_{-\frac{1}{2}}(k)$.

Appendix A: Conventions for operators and partition functions.

In this appendix we summarize our conventions for operator fields, partition functions and spin structures in the explicit setting of a Kawai-Lewellen-Tye (KLT) heterotic string model. For more details, see refs. [12] and [11].

Space-time coordinate field:

$$X^\mu(z,\bar{z}) \;=\; q^\mu - ik^\mu \log z - ik^\mu \log \bar{z} + i\sum_{n\neq 0}\frac{a_n^\mu}{n} z^{-n} + i\sum_{n\neq 0}\frac{\bar{a}_n^\mu}{n}\bar{z}^{-n} \tag{A.1}$$

$$X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \;\stackrel{\text{OPE}}{=}\; -\eta^{\mu\nu}\log(z-w) + \text{ c.c. } + \ldots \tag{A.2}$$

$$\langle\langle\, 1\,\rangle\rangle_{g-\text{loop}} = (\det\bar{\partial}_0)^{-D/2}(\det 2\pi\text{Im}\tau)^{-D/2}\;, \tag{A.3}$$

where τ is the period matrix (as given in ref. [21]) and the explicit expression for $\det\bar{\partial}_0$ can be found for example in ref. [22]. It is normalized to give plus one in the limit where all loops are pinched.

Majorana fermion field:

$$\psi^\mu(z) = \sum_n \psi_n^\mu z^{-n-1/2} \qquad\qquad \{\psi_n^\mu, \psi_m^\nu\} = \eta^{\mu\nu}\delta_{n+m,0}\;, \tag{A.4}$$

where the mode index n is integer (half odd integer) for Ramond (Neveu-Schwarz) boundary conditions. The anti-commutation relations are equivalent to the OPE

$$\psi^\mu(z)\psi^\nu(w) \;\stackrel{\text{OPE}}{=}\; \eta^{\mu\nu}\frac{1}{z-w} + \ldots\;. \tag{A.5}$$

When computing correlation functions we usually bosonize all fermion fields.

Bosonized complex fermion:

$$\phi(z)\phi(w) \stackrel{\text{OPE}}{=} \log(z-w) + \ldots \tag{A.6}$$

$$\langle\langle \prod_{i=1}^{N} e^{q_i\phi(z_i)} \rangle\rangle_{g-\text{loop}} = \tag{A.7}$$

$$\delta_{\sum_{i=1}^{N} q_i,0} (\det \bar{\partial}_0)^{-1/2} \prod_{i<j} (E(z_i,z_j))^{q_i q_j} \; \Theta\left[{}^{\alpha}_{\beta}\right] \left(\sum_{i=1}^{N} q_i \int^{z_i} \frac{\omega}{2\pi i} | \tau \right) \,,$$

where ω_μ is normalized to have period $2\pi i \delta_{\mu,\nu}$ around the cycle a_ν ($\mu, \nu = 1, \ldots, g$), $E(z,w)$ is the prime form (with short-distance behaviour $E(z,w) = (z-w) + \mathcal{O}(z-w)^2$) and we define

$$\begin{aligned}\Theta\left[{}^{\alpha}_{\beta}\right](z|\tau) \;=\; & \sum_{r\in\mathbf{Z}^g} \exp\left\{ 2\pi i \left[\frac{1}{2} \sum_{\mu,\nu=1}^{g} (r_\mu + \frac{1}{2} - \alpha_\mu)\tau_{\mu\nu}(r_\nu + \frac{1}{2} - \alpha_\nu) \right.\right. \\ & \left.\left. + \sum_{\mu=1}^{g} (r_\mu + \frac{1}{2} - \alpha_\mu)(z_\mu + \beta_\mu + \frac{1}{2}) \right] \right\} \,.\end{aligned} \tag{A.8}$$

Superghosts:

Our conventions for mode expansions, OPEs and “bosonization” of the superghosts β and γ are the standard ones [23]. We always remain inside the “little” algebra, i.e. excluding the zero mode of η and ξ. Our convention for the partition function is

$$\langle\langle \prod_{i=1}^{N} e^{q_i\phi(z_i)} \rangle\rangle_{g-\text{loop}} = \tag{A.9}$$

$$\delta_{\sum_{i=1}^{N} q_i - 2g+2,0} \; (\det \bar{\partial}_0)^{1/2} \; \prod_{i=1}^{N} (\sigma(z_i))^{-2q_i} \; \prod_{i<j} (E(z_i,z_j))^{-q_i q_j} \; \times$$

$$\prod_{\mu=1}^{g} \left(e^{-2\pi i (1/2+\beta_\mu)} \right) \; \left[\Theta\left[{}^{\alpha}_{\beta}\right] \left(-\sum_{j=1}^{N} q_j \int_{z_0}^{z_j} \frac{\omega}{2\pi i} + 2\Delta^{z_0} | \tau \right) \right]^{-1} \,.$$

This expression agrees with eq. (36) of ref. [24], except for the overall sign which differs in two regards: First there is the phase factor appearing at the beginning of the third line above, which is chosen in accordance with our definition of the spin structure summation coefficient given below. Second, the sign of the argument of the theta function is opposite to that of ref. [24], which amounts to a factor of minus one for odd spin structures. The

sign we quote above for the argument of the theta function is the one that is obtained when the correlation function is carefully constructed by sewing [25]. Hence it is the sign consistent with factorization. Our conventions for the differential σ and the Riemann class Δ^{z_0} are in accordance with ref. [21].

Reparametrization ghosts:

Our conventions for reparametrization ghosts follow ref. [23]. The normalization of the partition function is the standard one, and the explicit expressions can be found in refs. [7,26]. By definition the correlator

$$\langle\langle \left| \prod_{I=1}^{3g-3+N_{\mathrm{tot}}} (\eta_I|b) \prod_{i=1}^{N_{\mathrm{tot}}} c(z_i) \right|^2 \rangle\rangle \tag{A.10}$$

is positive definite.

Spin structure summation coefficient:

Finally the spin structure summation coefficient in eq. (2.2) is a product of one-loop summation coefficients which are given in accordance with ref. [11] by

$$\begin{aligned} C^{\boldsymbol{\alpha}_\mu}_{\boldsymbol{\beta}_\mu} &= \frac{1}{\prod_i M_i} \times \\ &\exp\left\{ -2\pi i \left[\sum_i (n_i^\mu + \delta_{i,0})(\sum_j k_{ij} m_j^\mu + s_i - k_{i0}) + \sum_i m_i^\mu s_i + \frac{1}{2} \right] \right\} , \end{aligned} \tag{A.11}$$

where $\mu = 1, \ldots, g$. The spin structure $\left[\begin{smallmatrix}\alpha_L \\ \beta_L\end{smallmatrix}\right]$ of the fermion labelled by $L \in \{1, \ldots, 32\}$ is related to the integers m_i^μ and n_i^μ through

$$\begin{aligned} \alpha_{L,\mu} &= \sum_i m_i^\mu \, (\boldsymbol{W}_i)_{(L)} \\ \beta_{L,\mu} &= \sum_i n_i^\mu \, (\boldsymbol{W}_i)_{(L)} \quad . \end{aligned} \tag{A.12}$$

For more details, see ref. [11].

Appendix B: Hermiticity properties of operators

In this appendix we summarize the hermiticity properties of various primary operators and provide some of the details in the derivation of eq. (7.3).

Φ_Δ		$\widehat{\Phi}_\Delta$	
X^μ		X^μ	
∂X^μ		$-\partial X^\mu$	
$e^{ik\cdot X}$		$e^{-ik\cdot X}$	
ψ^μ		ψ^μ	
$\Phi_{(L)}$	$L{=}1,\ldots,32$	$-\Phi_{(L)}$	$L{=}1,\ldots,32$
$e^{\mathbf{A}_L\Phi_{(L)}}$	$L{=}1,\ldots,32$	$e^{-\mathbf{A}_L\Phi_{(L)}}$	$L{=}1,\ldots,32$
$\Phi_{(33)}$		$\Phi_{(33)}$	
$\Phi_{(34)}$		$\Phi_{(34)} - 2\log$	
$e^{\mathbf{A}_L\Phi_{(L)}}$	$L{=}33,34$	$e^{\mathbf{A}_L\Phi_{(L)}}$	$L{=}33,34$
β		$-\beta$	
γ		γ	
η		$-\eta$	
ξ		$-\xi$	
$\partial\xi$		$\partial\xi$	
b		b	
c		c	

Table B1: Hermiticity properties of various primary conformal fields.

The hermitean conjugate field $\widehat{\Phi}_\Delta$ of a primary conformal field Φ_Δ of dimension Δ is defined by eq. (4.2) or eq. (4.3). In table B.1 we list various operator fields and their hermitean conjugates.

The cocycle operators $C_{(L)}$ appearing in the expression (7.2) and defined in detail in refs. [11,12] satisfy

$$\begin{aligned} \left(C_{(L)}\right)^\dagger &= \left(C_{(L)}\right)^{-1}\left(C_{\mathrm{gh}}^{(L)}\right)^2 \qquad \text{for} \;\; L = 1,\ldots,32 \\ \left(C_{(L)}\right)^\dagger &= C_{(L)} \exp\left\{-2\pi i \sum_{K=1}^{32} Y_{LK} J_0^{(K)}\right\} \qquad \text{for} \;\; L = 33, 34\ . \end{aligned} \tag{B.1}$$

Here $C_{\mathrm{gh}}^{(L)}$ is the cocycle operator involving the number operators of the reparametrization

ghosts and of the (η,ξ) system (excluding the (η,ξ) zero modes) and is given explicitly by

$$C_{\rm gh}^{(L)} = \exp\left\{-i\pi Y_{34,L}\left(N_{(\eta,\xi)} - N_{(b,c)} + N_{\bar{b},\bar{c}}\right)\right\} \ . \tag{B.2}$$

Using the hermiticity properties listed above we find that the hermitean conjugate of the spin field (7.2) is given by

$$\begin{aligned}\widehat{S}_{\mathbf{A}} = & \prod_{L=1}^{32}\left(C_{\rm gh}^{(L)}\right)^{2\mathbf{A}_L} \ \exp\left\{-2\pi i\sum_{L=1}^{32}\left(Y_{34,L}\mathbf{A}_{34} + Y_{33,L}\mathbf{A}_{33}\right)J_0^{(L)}\right\} \times \\ & S^{(34)}_{\mathbf{A}_{34}} S^{(33)}_{\mathbf{A}_{33}} S^{(32)}_{-\mathbf{A}_{32}} \ldots S^{(1)}_{-\mathbf{A}_1} \ .\end{aligned} \tag{B.3}$$

Here the factors appearing on the right hand side of the equality sign in the first line of eq. (B.3) can be ignored: Acting on any string state in the theory they give plus one. It is sufficient to verify this on some generic ground state $|\mathbf{B}\rangle$, created by the spin field operator $S_{\mathbf{B}}$, since non-zero mode creation operators contribute with integer values to $J_0^{(L)}$. One has to recall that $\mathbf{A}_{33}$ and $\mathbf{A}_{34}$ are always either both integer or both half-integer, that the number operators appearing in $C_{\rm gh}^{(L)}$ always take integer values, and finally one has to make use of the fact (see ref. [11]) that for a consistent choice of cocycles the following conditions hold

$$\begin{aligned}\phi_{33}[\mathbf{B}] &\equiv \sum_{L=1}^{32} Y_{33,L}\mathbf{B}_L + \epsilon\mathbf{B}_{33} - Y_{34,33}\mathbf{B}_{34} = \text{ integer} \\ \phi_{34}[\mathbf{B}] &\equiv \sum_{L=1}^{32} Y_{34,L}\mathbf{B}_L + Y_{34,33}\mathbf{B}_{33} - \epsilon\mathbf{B}_{34} = \text{ integer} \\ \phi_{33}[\mathbf{B}] &\overset{\text{MOD } 2}{=} \phi_{34}[\mathbf{B}]\end{aligned} \tag{B.4}$$

for any ground state $|\mathbf{B}\rangle$ existing in the theory, regardless of the value chosen for the parameter $\epsilon = \pm 1$.

If we finally reorder the individual spin fields in eq. (B.3) we obtain

$$\begin{aligned}\widehat{S}_{\mathbf{A}} &= \left(\prod_{L=1}^{32}\delta_{\mathbf{A}_L+\mathbf{B}_L,0}\right)\left(\prod_{L=33}^{34}\delta_{\mathbf{A}_L,\mathbf{B}_L}\right) S^{(1)}_{\mathbf{B}_1}\ldots S^{(34)}_{\mathbf{B}_{34}} \ e^{i\pi\mathbf{B}\cdot Y\cdot\mathbf{B}} \\ &= \left(\sigma_1^{(33)}\mathbf{C}^{-1}\right)_{\mathbf{AB}} S_{\mathbf{B}} \ ,\end{aligned} \tag{B.5}$$

where

$$\left(\mathbf{C}^{-1}\right)_{\mathbf{AB}} = \left(\prod_{L=1}^{33}\delta_{\mathbf{A}_L+\mathbf{B}_L,0}\right)\delta_{\mathbf{A}_{34},\mathbf{B}_{34}} \ e^{i\pi\mathbf{B}\cdot Y\cdot\mathbf{B}} \tag{B.6}$$

is the explicit expression for the matrix whose inverse is given by eq. (7.5).

Appendix C: Compatibility of the GSO projection and the map between $\mathcal{W}_{|\lambda\rangle}$ and $\mathcal{W}_{\langle\lambda|}$

In this appendix we show explicitly that the map (4.17) from $\mathcal{W}^{(q)}_{|\lambda\rangle}$ to $\mathcal{W}^{(q)}_{\langle\lambda|}$ is compatible with the GSO projection in the setting of a four-dimensional KLT heterotic string model [8,9]. In other words we want to show that given a vertex operator $\mathcal{W}^{(q)}_{|\lambda\rangle}$ creating a state $|\lambda\rangle$ satisfying the GSO conditions, the state $\chi_q|\lambda^{\mathrm{BPZ}}\rangle$, which is created by $\mathcal{W}^{(q)}_{\langle\lambda|}$, also satisfies the GSO conditions.

We first recall what is the form of the GSO projection conditions. We consider as usual all world-sheet fermions to be bosonized. Then the GSO conditions involve only the "momenta" $J_0^{(L)}$ of the resulting bosons, and it is sufficient to consider a generic ground state as created by the operator (7.1). If this state satisfies the GSO condition, so do all the states obtained from it by means of non-zero mode creation operators.

The GSO projection assumes the form (see ref. [11])

$$\mathbf{W}_i \cdot \mathbf{N}_{[\![\boldsymbol{\alpha}]\!]} - s_i(N^{(33)}_{[\![\alpha_{32}]\!]} - N^{(34)}_{[\![\alpha_{32}]\!]}) - \sum_j k_{ij} m_j - s_i - k_{0i} + \mathbf{W}_i \cdot [\![\boldsymbol{\alpha}]\!] \overset{\mathrm{MOD}\ 1}{=} 0 \ , \tag{C.1}$$

where our notation is that of ref. [11], except for the labelling of the complex fermions which is chosen in accordance with ref. [12] and the present paper, i.e. the left-moving fermions are labelled by $L = 1, \ldots, 22$, the internal right-moving ones by $L = 23, \ldots, 31$, the space-time related ones by $L = 32$ (the transverse) and $L = 33$ (the longitudinal), and the superghosts by $L = 34$.

Let us briefly recall that the sector (i.e. the set of boundary conditions for the fermions enumerated by $L = 1, \ldots, 32$) is specified by the 32-component vector

$$\boldsymbol{\alpha} = \sum_i m_i \boldsymbol{W}_i \ , \tag{C.2}$$

where the integer m_i takes values $0, 1, \ldots, M_i - 1$, M_i being the smallest integer such that $M_j \boldsymbol{W}_j$ (j not summed) is a vector of integer numbers. The number operators $N^{(L)}_{[\![\alpha_L]\!]}$ are related to the "momenta" $J_0^{(L)} = \mathbf{A}_L$ by

$$\begin{aligned} N^{(L)}_{[\![\alpha_L]\!]} &= \mathbf{A}_L - [\![1-\alpha_L]\!] + \frac{1}{2} \qquad \text{for} \quad L = 1, \ldots, 33 \\ N^{(34)}_{[\![\alpha_{32}]\!]} &= \mathbf{A}_{34} - [\![\alpha_{32}]\!] - \frac{1}{2} \ . \end{aligned} \tag{C.3}$$

As we have seen in section 7, given the state $|\mathbf{A}\rangle$ with $J_0^{(L)} = \mathbf{A}_L$, created by the spin field operator (7.2), the state $|\mathbf{A}^{\rm BPZ}\rangle$ created by the operator (7.3) has

$$J_0^{(L)} = \widetilde{\mathbf{A}}_L = \begin{cases} -\mathbf{A}_L & \text{for } L = 1, \ldots, 32 \\ +\mathbf{A}_L & \text{for } L = 33, 34 \end{cases} \quad . \tag{C.4}$$

This behaviour follows directly from the hermiticity properties of the various fields, as summarized in Appendix B. In eq. (7.3) it is encoded in the presence of the charge conjugation matrix $\mathbf{C}$ which changes sign on all the $\mathbf{A}_L$ except $\mathbf{A}_{34}$ and the factor $\sigma_1^{(33)}$ which changes sign on $\mathbf{A}_{33}$ only.

Thus we have to check if the GSO projection conditions (C.1) are invariant under the transformation $\mathbf{A}_L \to \widetilde{\mathbf{A}}_L$ given by (C.4). The situation is somewhat complicated by the fact that in general the states $|\mathbf{A}\rangle$ and $|\mathbf{A}^{\rm BPZ}\rangle$ do not reside in the same sector.

Let us denote by a tilde ($\widetilde{\ }$) the quantities pertaining to the state $|\mathbf{A}^{\rm BPZ}\rangle$. We want to show then that if the state $|\mathbf{A}\rangle$, residing in the sector $\boldsymbol{\alpha}$, satisfies eq. (C.1), then the state $|\mathbf{A}^{\rm BPZ}\rangle$, residing in the sector $\widetilde{\boldsymbol{\alpha}}$, satisfies

$$\mathbf{W}_i \cdot \widetilde{\mathbf{N}}_{[\![\widetilde{\boldsymbol{\alpha}}]\!]} - s_i(\widetilde{N}^{(33)}_{[\![\widetilde{\alpha}_{32}]\!]} - \widetilde{N}^{(34)}_{[\![\widetilde{\alpha}_{32}]\!]}) - \sum_j k_{ij}\widetilde{m}_j - s_i - k_{0i} + \mathbf{W}_i \cdot [\![\widetilde{\boldsymbol{\alpha}}]\!] \overset{\text{MOD } 1}{=} 0 \ . \tag{C.5}$$

We can actually do something more general, and for this we take the sum of eqs. (C.1) and (C.5). We will show that this sum is zero modulus one, this obviously implies that if eq. (C.1) is satisfied then also eq. (C.5) is, and viceversa. In summing the two equations we make use of the fact that the s_i are half-integers [8], that eq. (C.4) implies $N^{(34)}_{[\![\alpha_{32}]\!]} = \widetilde{N}^{(34)}_{[\![\widetilde{\alpha}_{32}]\!]}$ and $N^{(33)}_{[\![\alpha_{32}]\!]} = \widetilde{N}^{(33)}_{[\![\widetilde{\alpha}_{32}]\!]}$ and that the number operators always have integer eigenvalues. Thus by summing we obtain

$$\mathbf{W}_i \cdot \left(\mathbf{N}_{[\![\boldsymbol{\alpha}]\!]} + \widetilde{\mathbf{N}}_{[\![\widetilde{\boldsymbol{\alpha}}]\!]}\right) - \sum_j k_{ij}\,(m_j + \widetilde{m}_j) - 2k_{0i} + \mathbf{W}_i \cdot ([\![\boldsymbol{\alpha}]\!] + [\![\widetilde{\boldsymbol{\alpha}}]\!]) \overset{\text{MOD } 1}{\underset{?}{=}} 0 \ . \tag{C.6}$$

In order to verify this identity we need to find the relation between $\boldsymbol{\alpha}$ and $\widetilde{\boldsymbol{\alpha}}$. We claim that

$$\widetilde{m}_j \ = \ \begin{cases} 0 & \text{if } m_j = 0 \\ M_j - m_j & \text{otherwise} \end{cases} \ , \tag{C.7}$$

which is consistent with $0 \le m_j, \widetilde{m}_j \le M_j - 1$. The proof is simple: Let $L \in \{1, \ldots, 32\}$. Since the number operators $N^{(L)}_{[\![\alpha_L]\!]}$ take integer values, it follows from eq. (C.3) that the allowed values for $\mathbf{A}_L$ in the sector $\boldsymbol{\alpha} = \sum_j m_j \boldsymbol{W}_j$ are

$$\mathbf{A}_L \ = \ \tfrac{1}{2} - \sum_j m_j (\boldsymbol{W}_j)_{(L)} + (\text{integer}) \ . \tag{C.8}$$

Since $(\boldsymbol{W}_j)_{(L)} = w_{j,(L)}/M_j$ where $w_{j,(L)}$ is an integer satisfying $0 \le w_{j,(L)} \le M_j - 1$, we have

$$\mathbf{A}_L \overset{\text{MOD } 1}{=} \tfrac{1}{2} - \sum_j m_j \frac{w_{j,(L)}}{M_j} \ . \tag{C.9}$$

Likewise, in the sector $\widetilde{\boldsymbol{\alpha}}$ we have for $L = 1, \ldots, 32$

$$\widetilde{\mathbf{A}}_L \ = \ -\mathbf{A}_L \overset{\text{MOD } 1}{=} \tfrac{1}{2} - \sum_j \widetilde{m}_j \frac{w_{j,(L)}}{M_j} \ . \tag{C.10}$$

Comparing eqs. (C.9) and (C.10) we find the obvious solution $\widetilde{m}_j = -m_j \ (\text{mod} M_j)$ which is equivalent to (C.7).

It is worth noticing that if we sum the two equations

$$\mathbf{A}_L \overset{\text{MOD } 1}{=} \tfrac{1}{2} - [\![\alpha_L]\!] \qquad , \qquad \widetilde{\mathbf{A}}_L \ = \ -\mathbf{A}_L \overset{\text{MOD } 1}{=} \tfrac{1}{2} - [\![\widetilde{\alpha}_L]\!] \tag{C.11}$$

we get

$$0 \overset{\text{MOD } 1}{=} [\![\alpha_L]\!] + [\![\widetilde{\alpha}_L]\!] \ . \tag{C.12}$$

Thus, $[\![\alpha_L]\!] = [\![\widetilde{\alpha}_L]\!]$ only if the fermion labelled by L satisfies either Neveu-Schwarz boundary conditions ($[\![\alpha_L]\!] = [\![\widetilde{\alpha}_L]\!] = 1/2$) or Ramond boundary conditions ($[\![\alpha_L]\!] = [\![\widetilde{\alpha}_L]\!] = 0$).

Let us now return to the identity (C.6) that we were supposed to prove. Substituting eq. (C.7) and recalling that $M_j k_{ij} \overset{\text{MOD } 1}{=} 0$ [8], we find that the term $\sum_j k_{ij}(m_j + \widetilde{m}_j)$ cancels out. Next we recall that [8]

$$2(k_{0i} + k_{i0}) \overset{\text{MOD } 1}{=} 2k_{0i} \overset{\text{MOD } 1}{=} 2\boldsymbol{W}_i \cdot \boldsymbol{W}_0 \tag{C.13}$$

where $\mathbf{W}_0$ is the vector with all entries equal to 1/2. Substituting all this into eq. (C.6) and using eq. (C.3) we find that eq. (C.6) holds if and only if

$$\mathbf{W}_i \cdot ([\![\boldsymbol{\alpha}]\!] + [\![\widetilde{\boldsymbol{\alpha}}]\!] - [\![1 - \boldsymbol{\alpha}]\!] - [\![1 - \widetilde{\boldsymbol{\alpha}}]\!]) \underset{?}{\overset{\text{MOD } 1}{=}} 0 \ , \tag{C.14}$$

where, rather obviously, $[\![1 - \boldsymbol{\alpha}]\!]$ is a vector whose L'th component is $[\![1 - \alpha_L]\!]$. The equation (C.14) is indeed satisfied since

$$[\![\alpha_L]\!] + [\![\widetilde{\alpha}_L]\!] - [\![1 - \alpha_L]\!] - [\![1 - \widetilde{\alpha}_L]\!] \ = \ 0 \ . \tag{C.15}$$

Indeed, if $[\![\alpha_L]\!] = 0$ then also $[\![\widetilde{\alpha}_L]\!] = [\![1 - \alpha_L]\!] = [\![1 - \widetilde{\alpha}_L]\!] = 0$ and eq. (C.15) is trivially satisfied. Otherwise $[\![1 - \alpha_L]\!] = 1 - [\![\alpha_L]\!]$, $[\![1 - \widetilde{\alpha}_L]\!] = 1 - [\![\widetilde{\alpha}_L]\!]$ and by eq. (C.12) $[\![\alpha_L]\!] + [\![\widetilde{\alpha}_L]\!] = 1$ and again eq. (C.15) holds.

Thus eq. (C.6) is satisfied, and we have shown that if the vertex operator $\mathcal{W}^{(q)}_{|\lambda\rangle}$ creates a state in the GSO projected spectrum, i.e. a state that satisfies eq. (C.1), then so does the vertex operator $\mathcal{W}^{(q)}_{\langle\lambda|}$, and viceversa.

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